Graphs, Surfaces and Homology

Third Edition

Homology theory is a powerful algebraic tool that is at the centre of current research in topology and its applications. This accessible textbook will appeal to students interested in the application of algebra to geometrical problems, specifically the study of surfaces (such as sphere, torus, Möbius band, Klein bottle). In this introduction to simplicial homology – the most easily digested version of homology theory – the author studies interesting geometrical problems, such as the structure of two-dimensional surfaces and the embedding of graphs in surfaces, using the minimum of algebraic machinery and including a version of Lefschetz duality.

Assuming very little mathematical knowledge, the book provides a complete account of the algebra needed (abelian groups and presentations), and the development of the material is always carefully explained with proofs of all the essential results given in full detail. Numerous examples and exercises are also included, making this an ideal text for undergraduate courses or for self-study.

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Preface to the third edition

Since the second edition of this book, published in 1977, went out of print I have received what a less modest person might describe as fan-mail, lamenting the fact that a really accessible introduction to algebraic topology, through simplicial homology theory, was no longer available. Despite the lapse of years, and despite the multiplicity of excellent texts, nothing quite like this book has appeared to take its place. That is the reason why the book is being reprinted, newly and elegantly typeset, and with all the figures re-drawn, by Cambridge University Press. (I don't think TEX was even a gleam in the eye of that wonderful benefactor of mathematicians, Donald Knuth, when the first edition of the book was beautifully typed on a double keyboard mechanical typewriter by the then Miss Ann Garfield, now Mrs Ann Newstead.)

Of course, homology theory has advanced in the interim – not just the theory, but most importantly the multiplicity of applications and interactions with other areas of mathematics and other disciplines. Shape description, robotics, knot theory (Khovanov homology), algebraic geometry and theoretical physics – to name just a few areas – use topological ideas and in particular homology and cohomology. A quick search with an Internet search engine will turn up many references to applications. Since this book is intended for beginners, there is no pretence of being able to cover the recent developments and I have confined my 'updating' to correcting obvious errors, replacing references to other textbooks with more accessible modern ones, including additional references to the research literature and adding some comments where it seemed appropriate. The driving force behind the book remains to present a thoroughly accessible and self-contained introduction to homology theory via its most easily digested manifestation, simplicial homology.

I thank again those who helped with the first and second editions, namely, in alphabetical order, Ronnie Brown, Bill Bruce, Pierre Damphousse, Chris Gibson, Erwin Kronheimer, Hugh Morton, Ann Newstead, John Reeve, Stephen

Wilson and Shaun Wylie. I now thank Jon Woolf for his advice, and Cambridge University Press for taking on a new edition – and, of course, my students and those who wrote telling me that they liked my book.

A final word on literary manners, which change over the years. In re-reading the text I was conscious of the use of ‘he’ and ‘his’ when referring to ‘the reader’. At the time of writing the first edition, everyone understood that these meant ‘he or she’ and ‘his or her’, but we have become more sensitive to gender issues in the meantime. Rather than laboriously change all these by some circumvention or other let me state here, once for all, that I truly hope to interest female and male readers without distinction, and I always use the pronouns, when not referring to a specific person, to mean either gender.

Preface to the first edition

Topology is pre-eminently the branch of mathematics in which other mathematical disciplines find fruitful application. In this book the algebraic theory of abelian groups is applied to the geometrical and topological study of objects in euclidean space, by means of homology theory. Several books on algebraic topology contain alternative accounts of homology theory; mine differs from these in several respects.

Firstly the book is intended as an undergraduate text and the only mathematical knowledge which is explicitly assumed is elementary linear algebra. In particular I do not assume, in the main logical stream of the book, any knowledge of point-set (or 'general') topology. (There are a number of tributaries, not part of the main stream but I hope no less logical, about which more in a moment.) A reader who is familiar with the concept of a continuous map will undoubtedly be in a better position to appreciate the significance of homology theory than one who is not but nevertheless the latter will not be at a disadvantage when it comes to understanding the proofs.

The avoidance of point-set topology naturally imposes certain limitations (in my view quite appropriate to a first course in homology theory) on the material which I can present. I cannot, for example, establish the topological invariance of homology groups. A weaker result, sufficient nevertheless for our purposes, is proved in Chapter 5, where the reader will also find some discussion of the need for a more powerful invariance theorem and a summary of the proof of such a theorem.

Secondly the emphasis in this book is on low-dimensional examples – the graphs and surfaces of the title – since it is there that geometrical intuition has its roots. The goal of the book is the investigation in Chapter 9 of the properties of graphs in surfaces; some of the problems studied there are mentioned briefly in the Introduction, which contains an informal survey of the material of the book. Many of the results of Chapter 9 do indeed generalize to higher dimensions (and the general machinery of simplicial homology theory is available from earlier chapters) but I have confined myself to one example, namely the theorem that non-orientable closed surfaces do not embed in three-dimensional space. One of the principal results of Chapter 9, a version of Lefschetz duality, certainly generalizes, but for an effective presentation such a generalization needs cohomology theory. Apart from a brief mention in connexion with Kirohhoff's laws for an electrical network I do not use any cohomology here.

Thirdly there are a number of digressions, whose purpose is rather to illuminate the central argument from a slight distance, than to contribute materially to its exposition. (To change the metaphor, these are the tributaries mentioned earlier. Some of them would turn into deep lakes if we were to pursue them far.) The longest digression concerns planar graphs, and is to be found in Chapter 1, but there are others on such topics as Kirchhoff's laws (also Chapter 1), minimal triangulations, embedding and colouring problems (Chapter 2), orientation of vector spaces (Chapter 3) and a game due to J. H. Conway (Chapter 9). In addition, there are a good many examples which extend the material of the text in a more or less informal way.

Material which is not directly relevant to the logical development is indicated by a bold asterisk ★ at the beginning and the end of the material,† or occasionally by some cautionary words at the outset. Thus the digressions in question can be regarded as optional reading, but even the reader intent on getting to the results may find the more relaxed atmosphere of some of the starred items congenial. Occasional use is made there of concepts and theorems which are related to the material of the text, but whose full exposition would take us too far afield. As an example, the proof of Fáry's theorem on straightening planar graphs (Theorem 1.34) assumes several plausible (but profound) results of point-set topology. The inclusion of this theorem makes possible several useful discussions later in the book; I have been careful to warn the reader that these discussions, resting as they do on a result whose complete proof involves concepts not explained here, must remain to some extent 'informal'. Investigations of this kind, and others in which an admission of informality is made, are in fact reminiscent of the style of argument usually adopted in more advanced mathematical work. The tacit assumption is present, that the author (and hopefully the reader) could, if he were so minded, fill in all the details and make the arguments formal, but that to do so in print would be, in this conservation-conscious age, a misuse of paper.

The fourth way in which my book differs from others of its kind is more technical in nature. I use collapsing as a major tool, rather than the Mayer–Vietoris sequence, which appears in a minor role in Chapter 7. My reason for

† thus asterisks do not direct the reader to footnotes, the symbol for which is (as you see) †.

using collapsing is that it leads straight to the calculation of the homology groups of surfaces with only a minimal need for invariance properties.

The theory of abelian groups enters into homology theory in two ways. The very definition of homology groups is framed in terms of quotient groups, and many theorems are expressed by the exactness of certain sequences. The abelian group theory used in this book is gathered together in an Appendix, in which proofs will be found of all the results used in the text.

About three-quarters of the material in this book was included in a course of around forty lectures I gave to third year undergraduates at Liverpool University. Included in the forty lectures were a few given by the students themselves. Some of these were on topics which now appear as optional sections, while others took their subject matter from research articles. The material now included in the text and which I omitted from the course comprises some of the work on planar graphs, all the work on invariance, the homology sequence of a triple, a few of the applications in Chapter 9, and several small optional sections. There are many other ways of cutting down the material. For example it is possible to restrict attention entirely to two-dimensional simplicial complexes: to facilitate this suggestion I have tried to ensure that by judicious skipping in Chapter 3 all mention of complexes of dimension higher than two can be suppressed. Thereafter the results can simply be restricted to this special case, and in Chapter 9 most of the work is two-dimensional anyway. Alternatively, Chapters 5 and 7 can be omitted, together with optional material according to the taste (or distaste) of the lecturer. I think it would be completely mistaken to give a course in which the students sharpened their teeth on all the techniques, only to leave without a sniff at the applications: better to omit details from Chapters 1–8 in order to include at any rate a selection of material from Chapter 9.

I have kept two principles before me, which I think should apply to all university courses in mathematics, and especially ones at a fairly high level. Firstly the course should have something to offer for those students who are going to take the subject further – it should be a stepping-stone to 'higher things'. But secondly, and I think this is sometimes forgotten, the course should exhibit a piece of mathematics which is reasonably complete in itself, and prove results which are interesting for their own sake. I venture to hope that the study of graphs in surfaces is interesting for its own sake, and that such a study provides a good introduction to the powerful and versatile techniques of algebraic topology. Bearing in mind that a proportion of students will become schoolteachers, I have been careful to include a number of topics, such as planar graphs in Chapter 1, colouring theorems in Chapter 2 and 'Brussels Sprouts' in Chapter 9, which, suitably diluted and imaginatively presented, could stimulate the interest of schoolchildren.

I have found it impossible in practice to separate worked examples, which often have small gaps in the working, from exercises, which often have hints for solution. All these are gathered together under the heading of ‘Examples’, but to help the reader in selecting exercises I have marked by X those examples where a substantial amount is left to the reader to verify. Two X’s indicate an example that I found difficult. Some examples are starred: this indicates that their subject matter is outside the main concern of the book.

References to articles are given by the name of the author followed by the date of publication in brackets. A list of references appears at the end of the text. Some of the references within the text are to research articles and more advanced textbooks where the reader can find full details of subjects which cannot be explored here. Inevitably these references are often technical and not for the beginner unless he is very determined. I have also included among the references several books which, though they are not specifically referred to in the text, can be regarded as parallel reading.

There is a list of notation on page xvii. My only conscious departure from generally received notation is the use, in these days a little eccentric, of the declaration Q.E.D. to announce the end of a proof[†].

[†] For the present edition I have succumbed to the temptation to use the open square □ since this can flexibly indicate the end or the absence of a proof, and also the end of a statement whose proof preceded it.

Notation

I *Symbols* (Completely standard set-theoretic notation is not listed.)

Symbol	*Example*	*Use*	*Page*
juxtaposition	$s_p s_q$, KL	join	86
	K^r	r-skeleton of K	77
	fx	value of f at x	
	na	n integer, $a \in$ group	216
$\Rightarrow$		implies	
$\Leftrightarrow$		if and only if	
$\backslash$	$X \backslash Y$	difference of sets	
$\searrow^e$	$K \searrow^e L$	elementary collapse	89
$\searrow$	$K \searrow L$	collapse	89
/	B/A	quotient group	220
|	$\delta_1 \| \delta_2$	δ_1 is a factor of δ_2	231
( )	(vw)	edge	11
	$(v^0 \cdots v^p)$	p-simplex, p-chain	70, 100
	$K^{(r)}$	rth barycentric subdivision	135
(,)	(v, w)	edge	11
	(K, L)	pair	139
(; ,)	$(K; L_1, L_2)$	triad	159
(, ,)	(M, N, K)	triple	168
{ }	$\{z\}$, $\{z\}_K$, $\{z\}_{K,L}$	homology class	105, 114, 174
{ , }	$\{v, w\}$	edge	10
⟨ , ⟩	$\langle \gamma, c \rangle$	value of cochain on chain	36
⟨ ⟩	$\langle a \rangle$	subgroup generated by a	237

Symbol	*Example*	*Use*	*Page*
$\approx$	$M \approx N$	equivalence of closed surfaces	53
$\cong$	$A \cong B$	isomorphism of groups	217
#	$M \# N$	connected sum	54
$\bullet$	$\overset{\bullet}{s}_n$	boundary of simplex	71, 76
$\circ$	$\overset{\circ}{s}_n$	interior of simplex	71
$<$	$s_p < s_n$	face of	72
$\| \ \|$	$\|K\|$	underlying space of	74
	$\|A\|$	number of elements of	216
$-$	$\bar{s}_n$	closure of simplex	76
	$\overline{\Sigma}, \overline{\mathrm{St}}$	closure of subset or star	84, 85
	$\bar{b}$	coset	220
	$\overline{M}$	closed surface obtained from surface	188
$\frown$	$(v^0 \cdots \widehat{v^i} \cdots v^p)$	simplex with vertices $v^0 \cdots v^p$ except v^i	101
	$\widehat{\partial}$	relative boundary	113
	$\widehat{s}$	barycentre of	136
$\sim$	$z \sim z'$	homologous to	105
	$\widetilde{H}_0$	reduced homology group	104
	$\widetilde{i}_*$	reduced homomorphism	140
	$\widetilde{A}$	vector space from abelian group	233
	$\widetilde{f}$	linear map from homomorphism	234
$\sim$(mod 2)	$z \sim z'$(mod 2)	homologous mod 2 to	174
$\overset{L}{\sim}$	$z \overset{L}{\sim} z'$	homologous mod L	114
$\oplus$	$A \oplus B$	direct sum	222, 223
$\otimes$	$A \otimes B$	tensor product	179, 233
$'$	K'	barycentric subdivision	136
$''$	K''	second barycentric subdivision	183
$+$	V_i^+	region determined by V_i	183

II *Roman characters*

B	$B_p(K)$	boundary group	104, 124
	$B_p(K,L)$		114
	$B_p(K;2)$		174
	$B_p(K,L;2)$		176

Symbol	*Example*	*Use*	*Page*
C	$C_p(K)$	chain group	21, 99, 124
C	$C_p(K, L)$		113
	$C_p(K; 2)$		171
	$C_p(K, L; 2)$		176
	$C^p(G)$	cochain group	36
	CK	cone on K	145
Cl	$\mathrm{Cl}(\Sigma, K)$	closure	84
F	FS	free abelian group on S	219
$\mathscr{F}$	$\mathscr{F}(R)$	frontier of region	30
$GL(n, \mathbb{R})$		general linear group	98
H	$H_p(K)$	homology group	104, 124
	$H_p(K, L)$		114
	$H_p(K; 2)$		174
	$H_p(K, L; 2)$		176
	$\widetilde{H}_0(K), \widetilde{H}_0(K; 2)$	reduced homology group	104, 174
$\mathcal{H}$	$\mathcal{H}(x)$	Heawood number	64
i_*		homomorphism from inclusion	121
$\tilde{i}_*$		reduced homomorphism	140
Im	$\mathrm{Im}\, f$	image	104, 217
j, j_q		homomorphism forgetting a subcomplex	113
j_*		homomorphism from j	122
$\tilde{j}_*$		reduced homomorphism	140
Ker	$\mathrm{Ker}\, f$	kernel	104, 217
Lk	$\mathrm{Lk}(s, K)$	link	87
N		regular neighbourhood	183
P, kP		projective plane, k-fold projective plane	54
$\mathbb{Q}$		rational numbers	218
$\mathbb{R}^n$		real euclidean or vector space of dimension n	
S		sphere	54
S^n		n-sphere	148
St, $\overline{\mathrm{St}}$	$\mathrm{St}(s, K)$	star, closed star	85
T, hT		torus, n-fold torus	54
Tor	$\mathrm{Tor}(A, B)$	torsion product	179

Symbol	*Example*	*Use*	*Page*
V		subcomplex complementary to regular neighbourhood	183
w_p		Stiefel–Whitney class	178
Z	$Z_p(K)$	cycle group	104, 124
	$Z_p(K, L)$		114
	$Z_p(K; 2)$		174
	$Z_p(K, L; 2)$		176
$\mathbb{Z}$		additive group of integers	216
$\mathbb{Z}_k$		$\{0, 1, \ldots, k-1\}$ under addition modulo k	216
$\mathbb{Z}^n$		group of n-tuples of integers	218

III. *Greek characters*

α, α_p		homomorphism	129, 167
	$\alpha_p(K), \alpha_p(K, L)$	number of p-simplexes of $K, K \backslash L$	55, 91, 155
β, β_p		homomorphism	130, 168
	$\beta_p(K)$	Betti number	155
	$\beta_p(K, L)$		155
	$\widehat{\beta}_p(K)$	connectivity number	174
γ, γ_p		homomorphism	168
δ		coboundary	36
∂, ∂_p		boundary homomorphism	21, 101, 172
∂	∂M	boundary of surface M	187
$\widehat{\partial}, \widehat{\partial}_p$		relative boundary	113
∂_*		homomorphism from ∂	123
$\widetilde{\partial}_*$		reduced homomorphism from ∂	140
Δ		homomorphism	159
	$\mathscr{G}^\Delta$	triangular graph	32
ε		augmentation	25, 103, 172
μ		cyclomatic number	19
ρ		restriction homomorphism	116
ϕ		homomorphism	158
χ	$\chi(K)$	Euler characteristic	55, 155
	$\chi(K, L)$		155
ψ		homomorphism	159