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Peter Hilton and Jean Pedersen
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A MATHEMATICAL TAPESTRY

Demonstrating the Beautiful Unity of Mathematics

This easy-to-read book demonstrates how a simple geometric idea reveals fascinating connections and results in number theory, polyhedral geometry, combinatorial geometry, and group theory. Using a systematic paper-folding procedure, it is possible to construct a regular polygon with any number of sides. This remarkable algorithm has led to interesting proofs of certain results in number theory, has been used to answer combinatorial questions involving partitions of space, and has enabled the authors to obtain the formula for the volume of a regular tetrahedron in around three steps, using nothing more complicated than basic arithmetic and the most elementary plane geometry. All of these ideas, and more, reveal the beauty of mathematics and the interconnectedness of its various branches.

Detailed instructions, including clear illustrations, enable the reader to gain hands-on experience constructing these models and to discover for themselves the patterns and relationships they unearth.

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This book is dedicated
to the memory of
Martin Gardner
(1914–2010)

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Preface

This is a book of 17 chapters, each of which provides some arithmetic, some geometry, or some algebra. The basic ideas in each chapter we call ***threads***, and there are at least nine threads in this book – paper-folding threads, number-theory threads, polyhedral threads, geometry threads, algebra threads, combinatorial threads, symmetry threads, group-theory threads, and historical threads. So this book utilizes, exploits, and develops, by weaving these threads of a very different kind together, many parts of mathematics. At the end of this preface we will give a table showing how you might read this book in the very unlikely event that you are interested in just one of these threads.

Many of the chapters involve the construction of models and these take time and effort, but we believe that if you choose to carry out the constructions you will find the activity satisfying. As Benno Artmann, reviewing one of Pedersen's articles in *Mathematical Reviews*, said about the construction of the golden dodecahedron: "I tried a dodecahedron. It sits on my desk, looks nice and makes me feel like an artist." On the other hand, we understand that many of our readers won't want to construct these models, but we think that they can be appreciated without actually constructing them. Surely you yourselves have enjoyed eating a piece of cake that someone else baked; and, even though you didn't actually do the baking, you might be interested in its ingredients and how it came to be in its final shape. So we include the instructions for building the models for those of you who want to construct them and hope that our other readers will at least appreciate what goes into making them.

We have had the immense benefit of the cooperation of the artist Sylvie Donmoyer who has provided beautiful, highly illustrative pictures, Hans Walser who has provided the figures for Chapter 13, and Byron Walden who took responsibility for the proofs in Chapter 17.

In addition to the various threads, there is one technical feature of this book which we would like to mention. We adopt, when appropriate, the notation of using radian measure, writing π (without the word "radians" following it) instead of 180°. The advantages of adopting this notation will, we believe, become obvious

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to our readers when it is used; since it clearly emphasizes that the straight angles of $180^\circ (= \pi)$ at the edge of the strip of paper being folded play a special role in the geometry, and subsequently in the number theory – as they obviously do. A feature which we use throughout the book, to make it easier for the reader to spot when a new idea is being named, is to use ***bold italic*** print to alert you that we are introducing a technical term that will be used subsequently. On the other hand, we use *ordinary italic* print for emphasis. We – and our readers – have found this convention helpful in some of our previous publications.

The topics of this book were chosen because of their interconnectedness; and the aim of the book is to show how pursuing a single idea in mathematics can lead in many different directions forming a unified whole. We think this book should be of interest to bright high-school students and all other intelligent people with an interest in mathematics because it touches on so many fascinating aspects of mathematics and the people who do it. We give some highlights below for number-theorists and geometers.

For those interested in number theory there are at least two very significant theorems in this book. It is intriguing to think that these two striking theorems about numbers came about naturally by following the mathematics of paper-folding described in Chapter 2, which was motivated by the hexaflexagons introduced in Chapter 1. Just to whet your appetite we will tell you now that the first of these important theorems, the quasi-order theorem, enables one to determine for any given odd number $b \geq 3$, using an algorithm that involves only subtraction and division by the number 2, the smallest power k to which 2 must be raised in order that either $2^k - 1$, or $2^k + 1$ is exactly divisible by b. And it tells us whether the sign should be "$-$" or "$+$". Furthermore, the algorithm never uses any number larger than b itself. The proof of this theorem is the most delicate result we present, but it seems, in the context of our development, to be a perfectly natural result. It leads, indeed, to a proof that the Fermat number, F_5, which is $2^{32} + 1$, is not prime (see Chapter 7). It is, in fact, the smallest Fermat number to be composite. Our proof is based on an algorithm that uses only subtraction and division by 2, and involves no number larger than 641.

The second significant number-theoretic result occurs in Chapter 7. We call that result the coach theorem because the mathematical symbols in the statement of the theorem look like coaches on a train to the English author, and we yielded to his wording, since to have called it a "car theorem" (because Americans refer to cars on a train) didn't sound nearly so nice to either of the authors. In Chapter 17 the coach theorem enables us, by a logical extension of the quasi-order theorem, to determine for any given $b \geq 3$ the number of proper divisors of the number b. In other words it gives us the value of what is well-known among number-theorists as the Euler totient function of b, that we denote $\Phi(b)$ (as defined in Section 17.1). This result

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is obtained through repeated use of the algorithm used for the quasi-order theorem. We have described both of these theorems in terms of the number 2 but they both have generalizations involving a general positive number $t \geq 2$, which we also give in Chapter 17 for the benefit of those truly interested in number theory.

There are also results about divisibility that are quite counter-intuitive. For example, in Chapter 4 we take the basic number fact $7 \times 3 = 21$ and show that the two related number facts

$$7 \text{ divides } 21$$

and

$$3 \text{ divides } 21$$

have very different generalizations to arbitrary bases!

Readers who are especially interested in geometry will find here a completely systematic method for constructing arbitrarily good approximations to regular b-gons, for *any* $b \geq 3$; this is a result that we believe the Greeks and Gauss would surely have liked to know about. These constructions led one of the authors to discover a construction of braided polyhedra (Chapters 8 and 9) with remarkable geometric properties of their own. Surprisingly, these same braided polyhedra are useful in determining the number of unbounded regions created in space by the extended face planes of the Platonic solids (Chapter 14).

The following table suggests which chapters to read for those of you with an overpowering interest in one or another of the nine threads. It should not be assumed, however, that if a chapter is not mentioned in one of the threads, then the thread does not appear at all there – this table lists the chapters that have some substantial parts devoted to the topic mentioned. For example, you may note that all chapters are listed under symmetry; and you will find that some are of an intensely geometric nature, while in other chapters the symmetry is in the statement of number-theoretic results, and in yet other chapters the symmetry is only present subtly without being mentioned explicitly.

Thread	Chapters
paper-folding	2, 3, 7
number theory	2, 3, 4, 7, 17
polyhedral	5, 6, 8, 9, 10, 11, 12, 13, 14, 15
geometry	1, 2, 3, 5, 6, 8, 9, 10, 11, 12, 14, 15, 16
algebra	1, 2, 3, 4, 5, 7, 17
combinatorial	10, 12, 14
symmetry	1–17
group theory	9, 11, 13, 14, 16
historical	1, 10, 15, 16

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We realize that not all readers will be interested in reading highly technical proofs. So we have placed an asterisk by the titles of certain sections, where the mathematics gets more intense, to let you know you can, with impunity, skip all (or part of) those sections on first reading and go straight on to the examples in those sections, or to other concepts. In almost all cases you will still be able to understand the subsequent material without having digested the proofs.

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