ADVANCED MECHANICS AND GENERAL RELATIVITY

Aimed at advanced undergraduates with background knowledge of classical mechanics and electricity and magnetism, this textbook presents both the particle dynamics relevant to general relativity, and the field dynamics necessary to understand the theory.

Focusing on action extremization, the book develops the structure and predictions of general relativity by analogy with familiar physical systems. Topics ranging from classical field theory to minimal surfaces and relativistic strings are covered in a consistent manner. Nearly 150 exercises and numerous examples throughout the textbook enable students to test their understanding of the material covered. A tensor manipulation package to help students overcome the computational challenge associated with general relativity is available on a site hosted by the author. A link to this and to a solutions manual can be found at www.cambridge.org/9780521762458.

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For Lancaster, Lewis, and Oliver

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Preface

Classical mechanics, as a subject, is broadly defined. The ultimate goal of mechanics is a complete description of the motion of particles and rigid bodies. To find $\mathbf{x}(t)$ (the position of a particle, say, as a function of time), we use Newton's laws, or an updated (special) relativistic form that relates changes in momenta to forces. Of course, for most interesting problems, it is not possible to solve the resulting second-order differential equations for $\mathbf{x}(t)$. So the content of classical mechanics is a variety of techniques for describing the motion of particles and systems of particles in the absence of an explicit solution. We encounter, in a course on classical mechanics, whatever set of tools an author or teacher has determined are most useful for a partial description of motion. Because of the wide variety of such tools, and the constraints of time and space, the particular set that is presented depends highly on the type of research, and even personality of the presenter.

This book, then, represents a point of view just as much as it contains information and techniques appropriate to further study in classical mechanics. It is the culmination of a set of courses I taught at Reed College, starting in 2005, that were all meant to provide a second semester of classical mechanics, generally to physics seniors. One version of the course has the catalog title "Classical Mechanics II", the other "Classical Field Theory". I decided, in both instantiations of the course, to focus on general relativity as a target. The classical mechanical tools, when turned to focus on problems like geodesic motion, can take a student pretty far down the road toward motion in arbitrary space-times. There, the Lagrangian and Hamiltonian are used to expose various constants of the motion, and applying these to more general space-times can be done easily. In addition, most students are familiar with the ideas of coordinate transformation and (Cartesian) tensors, so much of the discussion found in a first semester of classical mechanics can be modified to introduce the geometric notions of metric and connection, even in flat space and space-time.

So my first goal was to exploit students' familiarity with classical mechanics to provide an introduction to the geometric properties of motion that we find in general relativity. We begin, in the first chapter, by reviewing Newtonian gravity, and simultaneously, the role of the Lagrangian and Hamiltonian points of view, and the variational principles that connect the two. Any topic that benefits from both approaches would be a fine vehicle for this first chapter, but given the ultimate goal, Newtonian gravity serves as a nice starting point. Because students have seen Newtonian gravity many times, this is a comfortable place to begin the shift from $L = \frac{1}{2}\, m\, v^2 - U$ to an understanding of the Lagrangian as a geometric object. The metric and its derivatives are introduced in order to make the "length-minimizing" role of the free Lagrangian clear, and to see how the metric dependence on coordinates can show up in the equations of motion (also a familiar idea).

Once we have the classical, classical mechanics reworked in a geometric fashion, we are in position to study the simplest modification to the underlying geometry – moving the study of dynamics from Euclidean flat space (in curvilinear coordinates) to Minkowski space-time. In the second chapter, we review relativistic dynamics, and its Lagrange and Hamiltonian formulation, including issues of parametrization and interpretation that will show up later on. Because of the focus on the role of forces in determining the dynamical properties of relativistic particles, an advertisement of the "problem" with the Newtonian gravitational force is included in this chapter. That problem can be seen by analogy with electrodynamics – Newtonian gravity is not in accord with special relativity, with deficiency similar in character to Maxwell's equations with no magnetic field component. So we learn that relativistic dynamics requires relativistic forces, and note that Newtonian gravity is not an example of such a force.

Going from Euclidean space in curvilinear coordinates to Minkowski space-time (in curvilinear coordinates, generally) represents a shift in geometry. In the third chapter, we return to tensors in the context of these flat spaces, introducing definitions and examples meant to motivate the covariant derivative and associated Christoffel connection. These exist in flat space(-time), so there is an opportunity to form a connection between tensor ideas and more familiar versions found in vector calculus. To understand general relativity, we need to be able to characterize space-times that are not flat. So, finally, in the fourth chapter, we leave the physical arena of most of introductory physics and discuss the idea of curvature, and the manner in which we will quantify it. This gives us our first introduction to the Riemann tensor and a bit of Riemannian geometry, just enough, I hope, to keep you interested, and provide a framework for understanding Einstein's equation. At the end of the chapter, we see the usual motivation of Einstein's equation, as an attempt to modify Newton's second law, together with Newtonian gravity, under the influence of the weak equivalence principle – we are asking: "under what conditions can the motion of classical bodies that interact gravitationally, be viewed as length-minimizing paths in a curved space-time?" This is Einstein's idea, if everything undergoes the same motion (meaning acceleration, classically), then perhaps that motion is a feature of space-time, rather than forces.

At this point in the book, an abrupt shift is made. What happened is that I was asked to teach "Classical Field Theory", a different type of second semester of classical mechanics geared toward senior physics majors. In the back of most classical mechanics texts, there is a section on field theory, generally focused on fluid dynamics as its end goal. I again chose general relativity as a target – if geodesics and geometry can provide an introduction to the motion side of GR in the context of advanced mechanics, why not use the techniques of classical field theory to present the field-theoretic (meaning Einstein's equation again) end of the same subject? This is done by many authors, notably Thirring and Landau and Lifschitz. I decided to focus on the idea that, as a point of physical model-building, if you start off with a second-rank, symmetric tensor field on a Minkowski background, and require that the resulting theory be self-consistent, you end up, almost uniquely, with general relativity. I learned this wonderful idea (along with most of the rest of GR) directly from Stanley Deser, one of its originators and early proponents. My attempt was to build up enough field theory to make sense of the statement for upper-level undergraduates with a strong background in E&M and quantum mechanics.

So there is an interlude, from one point of view, amplification, from another, that covers an alternate development of Einstein's equation. The next two chapters detail the logic of constructing relativistic field theories for scalars (massive Klein–Gordon), vectors (Maxwell and Proca), and second-rank symmetric tensors (Einstein's equation). I pay particular attention to the vector case – there, if we look for a relativistic, linear, vector field equation, we get E&M almost uniquely (modulo mass term). The coupling of E&M to other field theories also shares similarities with the coupling of field theories to GR, and we review that aspect of model-building as well. As we move, in Chapter 6, to general relativity, I make heavy use of E&M as a theory with much in common with GR, another favorite technique of Professor Deser. At the end of the chapter, we recover Einstein's equation, and indeed, the geometric interpretation of our second-rank, symmetric, relativistic field as a metric field. The digression, focused on fields, allows us to view general relativity, and its interpretation, in another light.

Once we have seen these two developments of the same theory, it is time (late in the game, from a book point of view) to look at the physical implications of solutions. In Chapter 7, we use the Weyl method to develop the Schwarzschild solution, appropriate to the exterior of spherically symmetric static sources, to Einstein's equation. This is the GR analogue of the Coulomb field from E&M,

and shares some structural similarity with that solution (as it must, in the end, since far away from sources, we have to recover Newtonian gravity), and we look at the motion of test particles moving along geodesics in this space-time. In that setting, we recover perihelion precession (massive test particles), the bending of light (massless test particles), and gravitational redshift. This first solution also provides a venue for discussing the role of coordinates in a theory that is coordinate-invariant, so we look at the various coordinate systems in which the Schwarzschild space-time can be written and its physical implications uncovered.

Given the role of gravitational waves in current experiments (like LIGO), I choose radiation as a way of looking at additional solutions to Einstein's equation in vacuum. Here, the linearized form of the equations is used, and contact is again made with radiation in E&M. There are any number of possible topics that could have gone here – cosmology would be an obvious one, as it allows us to explore non-vacuum solutions. But, given the field theory section of the book, and the view that Maxwell's equations can be used to inform our understanding of GR, gravitational waves are a natural choice.

I have taken two routes through the material found in this book, and it is the combination of these two that informs its structure. For students who are interested in classical mechanical techniques and ideas, I cover the first four chapters, and then move to the last three – so we see the development of Einstein's equation, its role in determining the physical space-time outside a spherically symmetric massive body, and the implications for particles and light. If the class is focused on field theory, I take the final six chapters to develop content. Of course, strict adherence to the chapters will not allow full coverage – for a field theory class, one must discuss geodesic and geometric notions for the punchline of Chapter 7 to make sense. Similarly, if one is thinking primarily about classical mechanics, some work on the Einstein–Hilbert action must be introduced so that the Weyl method in Chapter 8 can be exploited.

Finally, the controversial ninth chapter – here I take some relevant ideas from the program of "advanced mechanics" and present them quickly, just enough to whet the appetite. The Kerr solution for the space-time outside a spinning massive sphere can be understood, qualitatively and only up to a point, by analogy with a spinning charged sphere from E&M. The motion of test bodies can be qualitatively understood from this analogy. In order to think about more exotic motion, we spend some time discussing numerical solution to ODEs, with an eye toward the geodesic equation of motion in Kerr space-time. Then, from our work understanding metrics, and relativistic dynamics, combined with the heavy use of variational ideas throughout the book, a brief description of the physics of relativistic strings is a natural topic. We work from area-minimization in Euclidean spaces to area-minimization in Minkowski space-times, and end up with the standard equations of motion for strings.

I have made available, and refer to, a minimal `Mathematica` package that is meant to ease some of the computational issues associated with forming the fundamental tensors of Riemannian geometry. While I do believe students should compute, by hand, on a large piece of paper, the components of a nontrivial Riemann tensor, I do not want to let such computations obscure the utility of the Riemann tensor in geometry or its role for physics. So, when teaching this material, I typically introduce the package (with supporting examples, many drawn from the longer homework calculations) midway through the course. Nevertheless, I hope it proves useful for students learning geometry, and that they do not hesitate to use the package whenever appropriate.

A note on the problems in this book. There are the usual set of practice problems, exercises to help learn and work with definitions. But, in addition, I have left some relatively large areas of study in the problems themselves. For example, students develop the Weyl metric, appropriate to axially symmetric space-times, in a problem. The rationale is that the Weyl metric is an interesting solution to Einstein's equation in vacuum, and yet, few astrophysical sources exhibit this axial symmetry. It is an important solution, but exploring the detailed physics of the solution is, to a certain extent, an aside. In the end, I feel that students learn best when they develop interesting (if known) ideas on their own. That is certainly the case for research, and I think problems can provide an introduction to that process. In addition to practicing the techniques discussed in the text, working out long, involved, and physically interesting problems gives students a sense of ownership, and aids retention. Another example is the verification that the Kerr solution to Einstein's equation is in fact a vacuum solution. Here, too, a full derivation of Kerr is beyond the techniques introduced within the book, so I do not consider the derivation to be a primary goal – verification, however, is a must, and can be done relatively quickly with the tools provided. I have marked these more involved problems with a $*$ to indicate that they are important, but may require additional tools or time.

As appears to be current practice, I am proud to say that there are no new ideas in this book. General relativity is, by now, almost a century old, and the classical mechanical techniques brought to its study, much older. I make a blanket citation to all of the components of the Bibliography (found at the end), and will point readers to specific works as relevant within the text.

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I would like to thank my teachers, from undergraduate to postdoctoral: Nicholas Wheeler, Stanley Deser, Sebastian Doniach, and Scott Hughes, for their thoughtful advice, gentle criticism, not-so-gentle criticism, and general effectiveness in teaching me something (not always what they intended).

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Finally, Professor Deser introduced me to general relativity, and I thank him for sharing his ideas, and commentary on the subject in general, and for this book in particular. Much of the presentation has been informed by my contact with him – he has been a wonderful mentor and teacher, and working with him is always a learning experience, that is to say, a great pleasure.

[1] The *Oxford English Dictionary* defines a student to be "A person who is engaged in or addicted to study" – from that point of view, we are all students, so here I am referring to "younger" students, and specifically, younger students at Reed College.