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Newtonian gravity

The first job, in studying general relativity, must be to establish the predictions and limitations of Newtonian gravity, its immediate predecessor. So we begin by studying the usual mechanical formulations of central potentials associated with spherically symmetric central bodies. We shall study the orbital motion of "test particles", and in general, review the language and results of classical gravity.

1.1 Classical mechanics

The program of classical mechanics is to determine the trajectory of a particle or system of particles moving under the influence of some force. The connection between force and motion is provided by Newton's second law:

$$m\,\ddot{\mathbf{x}}(t) = \mathbf{F}, \tag{1.1}$$

supplemented by appropriate initial or boundary conditions. The forces are provided ($\mathbf{F}$ might be made up of a number of different forces) and we solve the above for $\mathbf{x}(t)$, from which any measurable quantity can be predicted.

As an approach to solving problems, Newton's second law can be difficult to work with. Given a generic force and multiple particles, a direct and complete solution proceeding from (1.1) is often unattainable. So we content ourselves with supplying less information than the full $\mathbf{x}(t)$ (sometimes, for example, we can easily find $\dot{\mathbf{x}}(t)$, but cannot proceed to $\mathbf{x}(t)$), or, we work on special classes of forces for which alternate formulations of the second law are tractable. It is with the latter that we will begin our work on Newtonian gravity – following a short review of the Lagrangian formulation of the equations of motion given a force derivable from a potential, we will see how the Lagrange approach can be used to simplify and solve for the trajectories associated with the Newtonian central

potential. From there, we will move on to the Hamiltonian formulation of the same problem.

1.2 The classical Lagrangian

Here we will define the Lagrangian formulation of the fundamental problem of classical mechanics: "Given a potential, how do particles move?" This section serves as a short review of Lagrangians in general, and the next section will specialize to focus on Keplerian orbits in the classical setting – if we are to understand the changes to the motion of particles in general relativity (GR), it behooves us to recall the motion in the "normal" case. Our ultimate goal is to shift from the specific sorts of notations used in introductory cases (for example, spherical coordinates), to a more abstract notation appropriate to the study of particle motion in general relativity.

As we go, we will introduce some basic tensor operations, but there will be more of this to come in Chapter 3. We just need to become comfortable with summation notation for now.

1.2.1 Lagrangian and equations of motion

A Lagrangian is the integrand of an action – while this is not the usual definition, it is, upon definition of action, more broadly applicable than the usual "kinetic minus potential" form. In classical mechanics, the Lagrangian leading to Newton's second law reads, in Cartesian coordinates:[1]

$$L = \underbrace{\frac{1}{2} m \left(\mathbf{v}(t) \cdot \mathbf{v}(t)\right)}_{\equiv T} - U(\mathbf{x}(t)), \tag{1.2}$$

where we view x, y and z as functions of a parameter t which we normally interpret as "time". The first term is the kinetic energy (denoted T), the second is the potential energy. Remember, the ultimate goal of classical mechanics is to find the trajectory of a particle under the influence of a force. Physically, we control the description of the system by specifying the particle mass, form of the force or potential, and boundary conditions (particle starts from rest, particle moves from point a to point b, etc.). Mathematically, we use the equations of motion derived from the Lagrangian, together with the boundary conditions, to determine the curve $\mathbf{x}(t) = x(t)\,\hat{\mathbf{x}} + y(t)\,\hat{\mathbf{y}} + z(t)\,\hat{\mathbf{z}}$ through three-dimensional space.

[1] I will refer to the "vector" (more appropriately, the coordinate differential is the vector) of coordinates as $\mathbf{x} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, and its time-derivative (velocity) as $\mathbf{v} = \frac{d\mathbf{x}}{dt}$.

Extremization of an action

The Euler–Lagrange equations come from the extremization, in the variational calculus sense, of the action:

$$S[\mathbf{x}(t)] = \int L(\mathbf{x}(t), \mathbf{v}(t))\, dt. \tag{1.3}$$

We imagine a path connecting two points $\mathbf{x}(0)$ and $\mathbf{x}(T)$, say. Then we define the dynamical trajectory to be the unique path that extremizes S. Suppose we have an arbitrary $\mathbf{x}(t)$ with the correct endpoints, and we perturb it slightly via $\mathbf{x}(t) \longrightarrow \mathbf{x}(t) + \boldsymbol{\eta}(t)$. In order to leave the physical observation of the endpoints unchanged, we require $\boldsymbol{\eta}(0) = \boldsymbol{\eta}(T) = 0$. The action responds to this change:

$$\begin{aligned} S[\mathbf{x}(t) + \boldsymbol{\eta}(t)] &= \int_0^T L(\mathbf{x}(t) + \boldsymbol{\eta}(t), \dot{\mathbf{x}}(t) + \dot{\boldsymbol{\eta}}(t))\, dt \\ &\approx \int_0^T \left(L(\mathbf{x}(t), \dot{\mathbf{x}}(t)) + \frac{\partial L}{\partial \mathbf{x}} \cdot \boldsymbol{\eta} + \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \dot{\boldsymbol{\eta}} \right) dt, \end{aligned} \tag{1.4}$$

where the second line is just the Taylor expansion of the integrand to first order in $\boldsymbol{\eta}$. The unperturbed value $S[\mathbf{x}(t)]$ is recognizable from the leading term in the integrand, all the rest is the change:

$$\Delta S = S[\mathbf{x}(t) + \boldsymbol{\eta}(t)] - S[\mathbf{x}(t)] = \int_0^T \left(\frac{\partial L}{\partial \mathbf{x}} \cdot \boldsymbol{\eta} + \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \dot{\boldsymbol{\eta}} \right) dt. \tag{1.5}$$

The small change $\boldsymbol{\eta}(t)$ is arbitrary, but once chosen, its time-derivative is fixed. We would like to write the integrand of (1.5) entirely in terms of the arbitrary trajectory perturbation, $\boldsymbol{\eta}(t)$, rather than quantities derived from this. We can use integration by parts on the second term to "flip" the t-derivative onto the L-derivative:

$$\int_0^T \frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \dot{\boldsymbol{\eta}}\, dt = \int_0^T \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \boldsymbol{\eta} \right) - \boldsymbol{\eta} \cdot \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{x}}} \right) \right) dt. \tag{1.6}$$

The first term, as a total time-derivative, gets evaluated at $t = 0$ and $t = T$ where $\boldsymbol{\eta}$ vanishes. We can use (1.6) to make the replacement under the integral in (1.5):

$$\frac{\partial L}{\partial \dot{\mathbf{x}}} \cdot \dot{\boldsymbol{\eta}} \longrightarrow -\boldsymbol{\eta} \cdot \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{x}}} \right), \tag{1.7}$$

and this leaves us with ΔS that depends on $\boldsymbol{\eta}(t)$ only:

$$\Delta S = \int_0^T \left(\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{x}}} \right) \right) \cdot \boldsymbol{\eta}\, dt. \tag{1.8}$$

Now extremization means $\Delta S = 0$, and the arbitrary value of $\boldsymbol{\eta}(t)$ allows us to set the term in parentheses equal to zero by itself (that's the only way to get $\Delta S = 0$ for arbitrary $\boldsymbol{\eta}(t)$).

As a point of notation, we use the variational derivative symbol δ to indicate that we have performed all appropriate integration by parts, so you will typically see (1.8) written as:

$$\delta S = \int_0^T \left(\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}} \right) \right) \cdot \delta \mathbf{x}\, dt, \tag{1.9}$$

where $\delta\mathbf{x}$ replaces $\boldsymbol{\eta}$ – this tells us that it is the variation with respect to $\mathbf{x}$ that is inducing the change in S. For actions that depend on more than one variable that can be varied, the notation makes it clear which one is being varied. In addition to this δS, $\delta\mathbf{x}$ shift, we will also refer to the Euler–Lagrange equations from variation with respect to $\mathbf{x}$ as:

$$\frac{\delta S}{\delta \mathbf{x}} = \left(\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{x}}} \right) \right), \tag{1.10}$$

the "variational derivative" of S with respect to $\mathbf{x}$. Extremization is expressed by $\delta S = 0$, or equivalently in this case, $\frac{\delta S}{\delta \mathbf{x}} = 0$.

Variation provides the ordinary differential equation (ODE) structure of interest, a set of three second-order differential equations, the Euler–Lagrange equations of motion:

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} - \frac{\partial L}{\partial \mathbf{x}} = 0.} \tag{1.11}$$

In Cartesian coordinates, with the Lagrangian from (1.2), the Euler–Lagrange equations reproduce Newton's second law given a potential U:

$$m\, \ddot{\mathbf{x}}(t) = -\nabla U. \tag{1.12}$$

The advantage of the action approach, and the Lagrangian in particular, is that the equations of motion can be obtained for any coordinate representation of the kinetic energy and potential. Although it is easy to define and verify the correctness of the Euler–Lagrange equations in Cartesian coordinates, they are not necessary to the formulation of valid equations of motion for systems in which Cartesian coordinates are less physically and mathematically useful.

The Euler–Lagrange equations, in the form (1.11), hold regardless of our association of $\mathbf{x}$ with Cartesian coordinates. Suppose we move to cylindrical coordinates $\{s, \phi, z\}$, defined by

$$x = s\, \cos\phi \qquad y = s\, \sin\phi \qquad z = z, \tag{1.13}$$

then the Lagrangian in Cartesian coordinates can be transformed to cylindrical coordinates by making the replacement for $\{x, y, z\}$ in terms of $\{s, \phi, z\}$ (and associated substitutions for the Cartesian velocities):

$$L(s, \phi, z) = L(\mathbf{x}(s, \phi, z)) = \frac{1}{2} m (\dot{s}^2 + s^2 \dot{\phi}^2 + \dot{z}^2) - U(s, \phi, z). \tag{1.14}$$

But, the Euler–Lagrange equations require no modification, the variational procedure that gave us (1.11) can be applied in the cylindrical coordinates, giving three equations of motion:

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} \\ 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} \\ 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z}. \end{aligned} \tag{1.15}$$

The advantage is clear: coordinate transformation occurs once and only once, in the Lagrangian. If we were to start with Newton's second law, we would have three equations with acceleration and coordinates coupled together. The decoupling of these would, in the end, return (1.15).

1.2.2 Examples

In one dimension, we can consider the Lagrangian $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k (x - a)^2$, appropriate to a spring potential with spring constant k and equilibrium spacing a. Then the Euler–Lagrange equations give:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m \ddot{x} + k (x - a) = 0. \tag{1.16}$$

Notice that for a real physical problem, the above equation of motion is not enough – we also need to specify two boundary conditions. We can phrase this choice in terms of boundaries in time at $t = t_0$ and $t = t_f$ (particle starts at 1 m from the origin at $t = 0$ and ends at 2 m from the origin at $t = 10$ s), or as an initial position and velocity (particle starts at equilibrium position with speed 5 m/s) – there are other choices as well, depending on our particular experimental setup.

In two dimensions, we can express a radial spring potential as:

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - \frac{1}{2} k (\sqrt{x^2 + y^2} - a)^2, \tag{1.17}$$

giving us two equations of motion:

$$\begin{aligned} 0 &= m\,\ddot{x} + \frac{k\,x\,(\sqrt{x^2+y^2}-a)}{\sqrt{x^2+y^2}} \\ 0 &= m\,\ddot{y} + \frac{k\,y\,(\sqrt{x^2+y^2}-a)}{\sqrt{x^2+y^2}}. \end{aligned} \tag{1.18}$$

Suppose we want to transform to two-dimensional polar coordinates via $x = s\,\cos\phi$ and $y = s\,\sin\phi$ – we can write (1.18) in terms of the derivatives of $s(t)$ and $\phi(t)$ and solve for $\ddot{s}$ and $\ddot{\phi}$ to get:

$$\begin{aligned} \ddot{s} &= -\frac{k}{m}\,(s-a) + s\,\dot{\phi}^2 \\ \ddot{\phi} &= -\frac{2\,\dot{\phi}\,\dot{s}}{s}. \end{aligned} \tag{1.19}$$

Actually performing this decoupling is not easy, and you should try the procedure for a few different coordinate system choices to appreciate the efficiency of the Lagrangian approach.

We can take advantage of the coordinate-independence of the Lagrangian to rewrite L directly in terms of $s(t)$ and $\phi(t)$, where it becomes:

$$L = \frac{1}{2}\,m\,(\dot{s}^2 + s^2\dot{\phi}^2) - \frac{1}{2}\,k\,(s-a)^2. \tag{1.20}$$

Then the Euler–Lagrange analogues of (1.15) become:

$$\begin{aligned} \frac{d}{dt}\,\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} &= m\,\ddot{s} - m\,s\,\dot{\phi}^2 + k\,(s-a) = 0 \\ \frac{d}{dt}\,\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} &= m\,(2\,s\,\dot{s}\,\dot{\phi} + s^2\,\ddot{\phi}) = 0, \end{aligned} \tag{1.21}$$

precisely (1.19), and well worth the effort of rewriting just the Lagrangian.

The Lagrangian written in polar coordinates (1.20) also provides insight into the motivation for using different coordinate systems. Notice that in the cylindrical case, the ϕ coordinate does not appear in the Lagrangian at all, only $\dot{\phi}$ shows up. Then we know from the Euler–Lagrange equations of motion that:

$$\frac{d}{dt}\,\frac{\partial L}{\partial \dot{\phi}} = 0 \longrightarrow \frac{\partial L}{\partial \dot{\phi}} = \text{constant of the motion.} \tag{1.22}$$

When possible, this type of observation can be useful in actually solving the equations of motion. Finding (or constructing) a coordinate system in which one or more of the coordinates do not appear is one of the goals of Hamilton–Jacobi theory.

Problem 1.1
Generate the equations of motion for the one-dimensional Lagrangian:

$$L = \frac{1}{2}\, m\, \dot{x}^2 - (A\, x + B)\,. \tag{1.23}$$

From the equations of motion (and the implicit definition of the potential), provide a physical interpretation for the constants A and B.

Problem 1.2
The Euler–Lagrange equations come from extremization of the action. So we expect the "true", dynamical trajectory to minimize (in this case) the value of $S = \int L\, dt$. For free particle motion, the Lagrangian is $L = \frac{1}{2}\, m\, v^2$ (in one dimension, for a particle of mass m). Suppose we start at $x(0) = 0$ and at time T, we end up at $x(T) = x_f$. The solution to the equation of motion is:

$$x(t) = \frac{x_f\, t}{T}. \tag{1.24}$$

(a) Compute $S = \int_0^T L\, dt$ for this trajectory.

(b) Any function that goes to zero at the endpoints can be represented in terms of the sine series, vanishing at $t = 0$ and $t = T$: $\sum \alpha_j\, \sin\left(\frac{j\pi\, t}{T}\right)$ by appropriate choice of the coefficients $\{\alpha_j\}_{j=1}^{\infty}$. So "any" trajectory for the particle could be represented by:

$$x(t) = \frac{x_f\, t}{T} + \sum_{j=1}^{\infty} \alpha_j\, \sin\left(\frac{j\,\pi\, t}{T}\right). \tag{1.25}$$

Find the value of the action $S = \int_0^T L\, dt$ for this arbitrary trajectory, show (assuming $\alpha_j \in \mathbb{R}$) that the value of the action for this arbitrary trajectory is greater than the value you get for the dynamical trajectory.

Problem 1.3
Take a potential in cylindrical coordinates $U(s)$ and write out the Euler–Lagrange equations (1.15) for the Lagrangian written in cylindrical coordinates (1.14). Verify that you get the same equations starting from Newton's second law in Cartesian coordinates and transforming to cylindrical coordinates, then isolating $\ddot{s}$, $\ddot{\phi}$ and $\ddot{z}$.

1.3 Lagrangian for $U(r)$

We want to find the parametrization of a curve $\mathbf{x}(t)$ corresponding to motion under the influence of a central potential. Central potentials depend only on a particle's distance from some origin, so they take the specific form: $U(x, y, z) = U(r)$ with $r^2 \equiv x^2 + y^2 + z^2$. We know, then, that the associated force will be directed, from

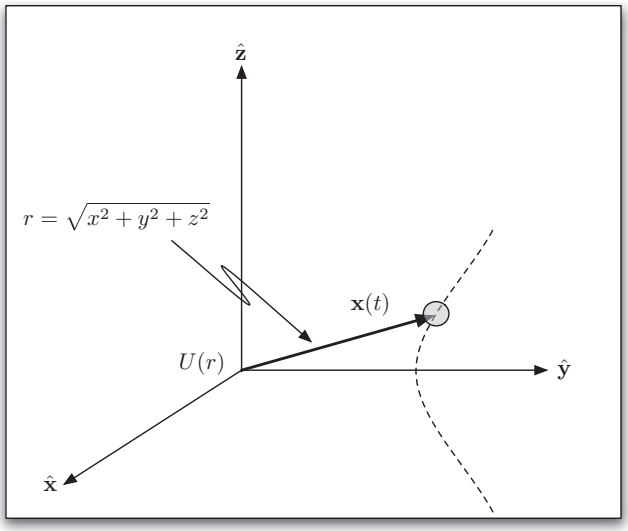

Figure 1.1 A particle trajectory. We want to find the curve $\mathbf{x}(t)$ from the equations of motion.

some origin, either toward or away from the particle (the force is $-\nabla U \sim \hat{\mathbf{r}}$, the usual result familiar from electrostatics, for example). Refer to Figure 1.1.

The Lagrangian for the problem is:

$$L = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(r). \tag{1.26}$$

We can move to spherical coordinates using the definition of $\{x, y, z\}$ in terms of $\{r, \theta, \phi\}$:

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta. \end{aligned} \tag{1.27}$$

Then the kinetic term in the Lagrangian will be made up of the following derivatives:

$$\begin{aligned} \dot{x}^2 &= \left(\dot{r} \sin\theta \cos\phi + r\,\dot{\theta} \cos\theta \cos\phi - r\,\dot{\phi} \sin\theta \sin\phi\right)^2 \\ \dot{y}^2 &= \left(\dot{r} \sin\theta \sin\phi + r\,\dot{\theta} \cos\theta \sin\phi + r\,\dot{\phi} \sin\theta \cos\phi\right)^2 \\ \dot{z}^2 &= \left(\dot{r} \cos\theta - r\,\dot{\theta} \sin\theta\right)^2, \end{aligned} \tag{1.28}$$

and inserting the coordinate and velocity forms into (1.26), we have:

$$\boxed{L = \frac{1}{2} m \left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2\right) - U(r).} \tag{1.29}$$

The equations of motion for this Lagrangian are the usual ones:

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\,\ddot{r} - m\,r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + \frac{\partial U}{\partial r} \\ 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m\,r^2\,\ddot{\theta} + 2\,m\,r\,\dot{r}\,\dot{\theta} - m\,r^2\,\sin\theta\,\cos\theta\,\dot{\phi}^2 \\ 0 &= \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = m\,r^2\,\sin^2\theta\,\ddot{\phi} + 2\,m\,r\,\sin^2\theta\,\dot{r}\,\dot{\phi} + 2\,m\,r^2\,\sin\theta\,\cos\theta\,\dot{\theta}\,\dot{\phi}. \end{aligned} \tag{1.30}$$

Now none of us would have gone this far without using some simplifying assumptions (angular momentum conservation, for instance), but the form of the above is interesting. Notice that there is only one term that we would associate with the physical environment (only one term involves the potential), the rest are somehow residuals of the coordinate system we are using.

More interesting than that is the structure of the equations of motion – everything that isn't $\ddot{X}$ (or $\frac{\partial U}{\partial r}$) looks like $f(r,\theta)\,\dot{X}\,\dot{Y}$ (here, $X, Y \in \{r, \theta, \phi\}$). That is somewhat telling, and says more about the structure of the Lagrangian and its quadratic dependence on velocities than anything else. Setting aside the details of spherical coordinates and central potentials, we can gain insight into the classical Lagrangian by looking at it from a slightly different point of view – one that will allow us to generalize it appropriately to both special relativity and general relativity. We will return to the central potential after a short notational aside.

1.3.1 The metric

And so, innocuously, begins our journey. Let's rewrite (1.26) in matrix–vector notation (we'll take potentials that are arbitrary functions of all three coordinates), the kinetic term is the beneficiary here:

$$L = \frac{1}{2}\,m\begin{pmatrix}\dot{x} & \dot{y} & \dot{z}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} - U(x,y,z). \tag{1.31}$$

We can make the move to spherical coordinates just by changing our curve coordinates from $\{x, y, z\}$ to $\{r, \theta, \phi\}$ and modifying the matrix:

$$L = \frac{1}{2}\,m\begin{pmatrix}\dot{r} & \dot{\theta} & \dot{\phi}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2\sin^2\theta\end{pmatrix}\begin{pmatrix}\dot{r}\\ \dot{\theta}\\ \dot{\phi}\end{pmatrix} - U(r,\theta,\phi). \tag{1.32}$$

But while it takes up a lot more space on the page, this is not so trivial a statement. Think of it this way – consider two points infinitesimally close together in the two

coordinate systems, the infinitesimal distance (squared) between the two points can be written in Cartesian or spherical coordinates:

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ ds^2 &= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \end{aligned} \tag{1.33}$$

These are just statements of the Pythagorean theorem in two different coordinate systems. The *distance* between the two points is the same in both, that can't change, but the representation is different.

These distances can also be expressed in matrix–vector form:

$$ds^2 = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{1.34}$$

and

$$ds^2 = \begin{pmatrix} dr & d\theta & d\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}. \tag{1.35}$$

The matrix form for infinitesimal length should look familiar (compare with (1.31) and (1.32)), and this isn't so surprising – velocity is intimately related to infinitesimal displacements. If we ask for the distance traveled along a curve parametrized by t in an infinitesimal interval dt, the answer is provided by:

$$ds = \sqrt{\left(\frac{dx}{dt}\,dt\right)^2 + \left(\frac{dy}{dt}\,dt\right)^2 + \left(\frac{dz}{dt}\,dt\right)^2} = \sqrt{\mathbf{v}\cdot\mathbf{v}}\,dt. \tag{1.36}$$

We call the matrix in (1.34) and (1.35) the *metric*. It is often represented not as a matrix but as a "second-rank tensor" and denoted $g_{\mu\nu}$. It tells us, given a coordinate system, how to measure distances. In classical mechanics, we usually go the other way around, as we have done here – we figure out how to measure distances in the new coordinates and use that to find $g_{\mu\nu}$ (actually, we rarely bother with the formal name or matrix, just transform kinetic energies and evaluate the equations of motion).

Label the vectors appearing in (1.34) and (1.35) dx^μ, so that the three components associated with $\mu = 1, 2, 3$ correspond to the three components of the vector:

$$dx^\mu \doteq \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}, \tag{1.37}$$