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Induced Representations of Locally Compact Groups

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We dedicate this book to our wives for their lifetime of support and exceptional patience during the preparation of the manuscript.

Contents

Preface

Locally compact groups arise in many diverse areas of mathematics, the physical sciences, and engineering and the presence of the group is usually felt through unitary representations of the group. This observation underlies the importance of understanding such representations and how they may be constructed, combined, or decomposed. Of particular importance are the irreducible unitary representations. In the middle of the last century, G. W. Mackey initiated a program to develop a systematic method for identifying all the irreducible unitary representations of a given locally compact group G. We denote the set of all unitary equivalence classes of irreducible unitary representations of G by $\widehat{G}$. Mackey's methods are only effective when G has certain restrictive structural characteristics; nevertheless, time has shown that many of the groups that arise in important problems are appropriate for Mackey's approach. The program Mackey initiated received contributions from many researchers with some of the most substantial advances made by R. J. Blattner and J. M. G. Fell. Fell's work is particularly important in studying $\widehat{G}$ as a topological space. At the core of this program is the inducing construction, which is a method of building a unitary representation of a group from a representation of a subgroup.

The main goal of this book is to make the theory of induced representations accessible to a wider audience. As the book progresses, we provide a large number of examples to illustrate the theory. A few particular groups reappear at various stages in the development of the material as more and more can be said about them.

We have written the book with the assumption that the reader will be familiar with the basics of harmonic analysis, the theory of unitary representations, and C^*-algebras. In the first chapter, we have gathered together the components most necessary for the main body of the book. We present these basic results, largely without proof, to orient the reader and establish notation. We recognize that not all readers will be completely familiar with all that is quickly covered in

Chapter 1, but we believe that the book can still serve as a useful reference for such readers mainly due to the variety of worked examples. Graduate students learning about induced representations here will want to have the standard references we mention in the first chapter at hand as they work through the later chapters.

If H is a closed subgroup of a locally compact group G and π is a representation of H, then the induced representation $\mathrm{ind}_H^G\,\pi$ is a representation of G. (Throughout this book, all representations of groups are unitary representations so we typically drop the word unitary.) In Chapter 2, we first define $\mathrm{ind}_H^G\,\pi$ in the case where H is an open subgroup of G. In that case, the construction of the Hilbert space on which $\mathrm{ind}_H^G\,\pi$ acts is particularly easy and the reader can concentrate on the algebraic manipulations to develop an intuitive feel for the inducing construction. Moreover, this enables us to quickly get to results of substance applicable to discrete groups such as the free group on two generators where, in Example 2.15, we construct a family of irreducible representations. After defining the induced representation in general, we provide several of the commonly used realizations and simplifications that occur in special cases such as inducing from the normal factor in a semidirect product group. Much of the rest of Chapter 2 is devoted to establishing the basic computational properties of the inducing construction, such as the vital induction in stages theorem.

The pivotal theorem of this book is the imprimitivity theorem, which is established in Chapter 3. Again, we prove it first in the open subgroup case, where the analytical details are straightforward, to illustrate the main strategy of the proof. Our proof in the general case is an elaboration of a proof given by Ørstedt [120].

With the imprimitivity theorem available, in Chapter 4 we turn to developing the systematic procedure, known as Mackey analysis, for constructing $\widehat{G}$ for a given locally compact group G. In order for this procedure to work, G must have a closed normal subgroup N such that $\widehat{N}$ is understood, the orbit structure under the action of G on $\widehat{N}$ must be well behaved, and stability subgroups (considered as subgroups of G/N) arising in this action must have a well-understood representation theory. The concepts involved simplify when N is abelian and simplify even more when G is the semidirect product of an abelian N and a group H acting on N.

We begin Chapter 4 by developing Mackey analysis for groups having an abelian subgroup of finite index; without loss of generality, we can take the abelian subgroup to be normal. The value in looking at this elementary case is that the role of the orbit structure in the dual of N becomes clear. With that in mind, we turn to the general situation of a closed normal abelian subgroup

N of G and carefully study the orbit space formed by G acting on $\widehat{N}$. There is a technical concept called Mackey compatibility for the subgroup N within G. It is actually fairly easy to recognize whether a particular N is a Mackey-compatible subgroup of a given G. When this happens, Theorem 4.27 provides a parametrization of $\widehat{G}$. The objects appearing in this parametrization are easiest to deal with when G splits as a semidirect product of the normal abelian subgroup N and another locally compact group H acting on N. Indeed, we are now able to provide two sections of examples where the analysis works perfectly. We believe that these worked out examples will be one of the valuable aspects for many readers. We also introduce some examples to illustrate the limitations when the abelian factor is not Mackey compatible in a semidirect product.

If G has a substantial closed normal abelian subgroup N but does not split as a semidirect product, then so-called cocycle representations must be used in the analysis. We briefly present the details and an illustrative example.

The final section of Chapter 4 deals with Mackey analysis in the case that the relevant closed normal subgroup N is not abelian. The treatment necessarily requires greater sophistication, with C^*-algebraic techniques and the orbit structure of actions on non-Hausdorff topological spaces. Nevertheless, Theorem 4.65 is established as the generalization of Theorem 4.27 to the case of a nonabelian N.

Tools for studying the topological structure of the dual space $\widehat{G}$, in those cases where Mackey analysis is successful, are developed in Chapter 5. The presentation of the main theorems follows the original proofs due to Fell. L. W. Baggett made significant contributions to understanding the topology of dual spaces and our treatment of generalized motion groups follows [2]. We hope that the extensive number of examples in Chapter 5 will contribute to more researchers taking advantage of the topological tools available in studying dual spaces.

Chapters 6 and 7 illustrate some of the different ways in which the theory of induced representations and knowledge of the topology of $\widehat{G}$ can be used to investigate other mathematical phenomena. We have included some topics from areas in which we have been involved personally, so these chapters certainly do not represent even a major sampling of the varied implications of the content of the earlier chapters. Chapter 6 is devoted to an exploration of topological versions of Frobenius properties generalizing the Frobenius reciprocity theorems of finite and compact groups. Chapter 7 explores the asymptotic behavior of the coefficient functions of the irreducible representations of motion groups and methods for constructing projections in the Banach $*$-algebra $L^1(G)$. In both these applications, we exploit the explicit structure of induced representations.

Chapters 1 to 4 are most useful for the researcher wishing to learn the basic techniques of induced representations and applying them to construct the duals of particular groups, while later chapters are intended more for specialists and to give an indication of the varied applications. For this reason, the exposition is more expansive in the first part of the book. For a graduate course in representation theory, a portion dedicated to induced representations could be supported by Chapters 2, 3, and 4.

Some of the topics in this book have, of course, been covered in other monographs. For example, Mackey's [105] provides the core of his theory while [104] and [106] are overviews which draw deep connections between the theory of induced representations and other areas of science. The books by Gaal [57], Barut and Raczka [15], and Fabec [39] each introduce some of the basic theory of induced representations and each has its own areas of focus. We have been very much influenced by the well-paced book by Folland [55] and the monumental volumes of Fell and Doran [53, 54]. In terms of level, this book lies between [55] and [53, 54]. The reader who is familiar with [55] can move quickly through our first three chapters, perhaps picking out some topics of Chapter 2 that are not touched on in [55]. However, much of the rest of our book is beyond the scope of [55]. Fell and Doran [53, 54] develop the general theory of Banach $*$-algebraic bundles from which the core theorems of this book can be extracted, but the task can be daunting for those new to the area. We have chosen to keep the focus clearly on the representation theory of locally compact groups and there are significant parts of this book which have not appeared in any monograph.

There are important classes of locally compact groups where either Mackey analysis is not effective or other methods provide more detailed information. Harish-Chandra, working in parallel with Mackey, developed a comprehensive approach to the representation theory of semisimple Lie groups. An excellent introduction to this theory is Knapp [93]. Kirillov showed that there is a bijective correspondence between the coadjoint orbits in the vector space dual of the Lie algebra of a connected and simply connected nilpotent Lie group G and $\widehat{G}$. This forms the basis for a detailed harmonic analysis on nilpotent Lie groups. Often the Kirillov construction and Mackey analysis can be used together in the study of a nilpotent Lie group. Corwin and Greenleaf [33] provides an introduction to the representation theory of nilpotent Lie groups.

Portions of the material in this book have been used by one or the other of us in graduate courses or seminars at the University of Paderborn, Technical University of Munich, the University of Saskatchewan, or Dalhousie University. We are grateful to those who attended these lectures for their feedback. The desire to formulate a more comprehensive manuscript on induced

representations grew out of our long-time collaborations. Our visits back and forth across the Atlantic for research purposes and the writing of this book have been supported by grants from NSERC Canada, the University of Paderborn, and Dalhousie University

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