Cambridge University Press
978-0-521-76226-7 - Induced Representations of Locally Compact Groups
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Excerpt
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Basics

This chapter provides the definitions and many of the basic properties of the objects of abstract harmonic analysis, including locally compact groups, their representations, and various algebras associated with both the groups and their representations. Besides providing the notational conventions used throughout the book, the necessary concepts are organized in a manner useful for the development of the theory of induced representations. For most of the propositions and theorems, we do not provide proofs as these are generally known and accessible in existing monographs. In Section 1.8, we provide a brief guide to the existing literature for the reader who seeks a more comprehensive treatment of a topic. However, we do provide full proofs in Section 1.3, which contains the tools for analysis on coset spaces that may not be so well known but are essential in defining induced representations and proving many of the key theorems.

1.1 Locally compact groups

A topological group G is a set with the structure of both a group and a topological space such that the group product is a continuous map from $G \times G$ into G and the group inverse is continuous on G. The group product of x and y in G will be denoted multiplicatively as xy and the inverse of x is x^{-1} except in a few specific cases such as the group of integers or the real numbers. In general, the identity element is denoted by e. If $y \in G$ is fixed, then each of the maps $R_y : x \to xy$, $L_y : x \to y^{-1}x$, and $x \to x^{-1}$ are homeomorphisms of G.

When the topology on G is Hausdorff and locally compact, we call G a *locally compact group*. The general theory of abstract harmonic analysis is highly developed for locally compact groups and this is the class of groups we are interested in here. We will use uppercase Latin letters such as G, H, N, K, or A to denote locally compact groups and lowercase Latin letters to denote their elements with exceptions such as f and g which will denote functions as necessary.

Fix a locally compact group G. For $A, B \subseteq G$, let $AB = \{xy : x \in A, y \in B\}$ and $A^{-1} = \{x^{-1} : x \in A\}$. The set A is called *symmetric* if $A^{-1} = A$. Also, for a natural number k, let $A^k = \{x_1 \cdots x_k : x_j \in A, 1 \leq j \leq k\}$. If A and B are compact, then so is AB, and if one of the sets A or B is open, then so is AB. If A is compact and B is closed, then AB and BA are closed.

If X is a locally compact Hausdorff space, a positive Borel measure μ on X is called *regular* if, for any Borel subset E of X,

$$\begin{aligned}\mu(E) &= \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\}\\ &= \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}.\end{aligned}$$

A complex Borel measure ν on X is regular if its total variation $|\nu|$ is regular.

A *Radon measure* on X is a positive Borel measure on X such that $\mu(K) < \infty$, for any compact set $K \subseteq X$,

$$\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\},$$

for any Borel subset E of X, and, for every open $U \subseteq X$,

$$\mu(U) = \sup\{\mu(K) : K \subseteq U,\ K \text{ compact}\}.$$

If μ is a σ-finite Radon measure, then μ is regular.

Returning to a locally compact group G, the existence of a translation-invariant Radon measure on G is of fundamental importance. A Borel measure μ on G is called *left invariant* (respectively, *right invariant*) if $\mu(xE) = \mu(E)$ (respectively, $\mu(Ex) = \mu(E)$), for any $x \in G$ and Borel subset E of G. The existence and uniqueness of a nonzero left-invariant Radon measure for G is sufficiently significant that we formulate the statement carefully.

Theorem 1.1 *Let G be a locally compact group. Then there exists a nonzero left-invariant Radon measure μ_G. It satisfies $\mu_G(U) > 0$ for any nonempty open subset U of G. If ν is any nonzero left-invariant Radon measure on G, then there is a constant $c > 0$ such that $\nu = c\mu_G$.*

Such a measure μ_G is called a *left Haar measure* on G. It is understood that a choice has been made out of the family $\{c\mu_G : c > 0\}$. Usually this choice is not made explicit, but if there is a distinguished compact neighborhood V of e, we may assume that $\mu(V) = 1$. For example, if G is compact itself, we assume

$\mu_G(G) = 1$. If G is an infinite group equipped with the discrete topology, we may assume that $\mu_G(\{e\}) = 1$. Then μ_G is simply a counting measure (use the left invariance).

Of course there also exists a right-invariant Radon measure on G with the same kind of uniqueness, a right Haar measure. In fact, the homeomorphism $x \to x^{-1}$ interchanges left and right Haar measures ($\nu(E) = \mu_G(E^{-1})$, for all Borel subsets E of G, defines a right Haar measure on G). We have chosen to use consistently left Haar measures.

Whenever we are working with a locally compact group G, we will assume without mentioning it that a left Haar measure is chosen. Actually, we will rarely use the notation μ_G. If A is a measurable subset of G, then $|A|$ will denote $\mu_G(A)$.

If $f \in C_c(G)$, the space of continuous complex-valued functions on G with compact support, then f is integrable with respect to μ_G and we usually write $\int_G f(x)d\mu_G(x)$ simply as $\int_G f(x)dx$. Indeed, we use the same simplification for any kind of function f on G (non-negative measurable, Haar integrable, or even vector-valued versions of integrability) for which $\int_G f(x)dx$ makes sense. The left invariance of μ_G implies that $\int_G f(yx)dx = \int_G f(x)dx$, for any $y \in G$. Note that if $f \in C_c^+(G) = \{g \in C_c(G) : g \geq 0\}$ and $f \neq 0$, then $\int_G f(x)dx > 0$.

On the other hand, if we fix a $y \in G$ and define a new measure ν by $\nu(E) = \mu(Ey)$, for all Borel subsets E of G, then ν is a left-invariant Radon measure on G that is positive on nonempty open sets. Thus, there is a positive constant $\Delta_G(y)$ so that $\mu(Ey) = \nu(E) = \Delta_G(y)\mu_G(E)$, for every Borel subset E of G. This gives a change of variables formula to use in integrals:

$$\int_G f(x)dx = \Delta_G(y) \int_G f(xy)dx,$$

for any function f where the integral makes sense and for any $y \in G$.

Letting y vary, $y \to \Delta_G(y)$ is a continuous homomorphism of G into $\mathbb{R}^+$, the multiplicative group of positive real numbers. It is called the *modular function* of G. The modular function enables a change of variables by inversion:

$$\int_G f(x^{-1})dx = \int_G f(x)\Delta_G(x^{-1})dx.$$

If $\Delta_G \equiv 1$ on G, that is, if every left Haar measure is also right invariant, then G is called *unimodular*. Of course, if G is abelian, then right and left translations are the same, and if G has the discrete topology, then the counting measure is both left and right invariant. Also, if G is compact, then $\Delta_G(G)$ is a compact subgroup of $\mathbb{R}^+$, so it must be trivial. Thus, each of these classes of groups, abelian, discrete, or compact, is contained in the class of all unimodular groups. Nevertheless, one frequently encounters nonunimodular groups, and the modular function and functions related to it play an important role in later chapters. For unimodular groups, the left Haar measure is also right invariant and we usually just refer to the *Haar measure* rather than the left Haar measure.

The Lebesgue space, $L^p(G, \mu_G)$, is denoted simply $L^p(G)$ for $1 \leq p \leq \infty$. Left and right translation of L^p-functions is continuous: given $f \in L^p(G)$ and $\epsilon > 0$, there exists a neighborhood U of e in G such that $\|L_y f - f\|_p < \epsilon$ and $\|R_y f - f\|_p < \epsilon$ for all $y \in U$. For $f \in L^1(G)$ and $g \in L^p(G)$, the integral $\int_G f(y)g(y^{-1}x)\,dy$ exists for almost all x. Therefore one can define

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dy, \tag{1.1}$$

for any $x \in G$ such that the right-hand side of (1.1) exists. The resulting measurable function, $f * g$, is called the *convolution* of f and g. Then $f * g \in L^p(G)$ and $||f * g||_p \leq ||f||_1||g||_p$.

When $p = 1$, this implies that $L^1(G)$, equipped with convolution as multiplication, is a Banach algebra. For $f \in L^1(G)$ and $x \in G$, let

$$f^*(x) = \Delta_G(x^{-1})\overline{f(x^{-1})}. \tag{1.2}$$

Then $||f^*||_1 = ||f||_1$ and $f \to f^*$ is an involution on $L^1(G)$.

Proposition 1.2 *Let G be a locally compact group. Then $L^1(G)$ is a Banach $*$-algebra when equipped with the convolution* (1.1) *and involution* (1.2).

If G is nondiscrete, $L^1(G)$ does not have an identity. However, it always possesses a two-sided approximate identity in $C_c(G)$, which can be constructed as follows. Let $\mathcal{U}$ be a neighborhood basis at e in G, and for each $U \in \mathcal{U}$ choose $f_U \in C_c^+(G)$ such that $f(x^{-1}) = f(x)$ for all $x \in G$, $\int_G f(x)dx = 1$, and with compact support contained in U. Then, for any $g \in L^p(G)$, $1 \leq p < \infty$, $\|f_U * g - g\|_p \to 0$ and $\|g * f_U - g\|_p \to 0$ as $U \to \{e\}$.

The Hilbert space structure of $L^2(G)$ is also fundamental. The inner product on $L^2(G)$ is given by

$$\langle f, g\rangle = \int_G f(x)\overline{g(x)}dx,$$

for $f, g \in L^2(G)$.

If X is a locally compact Hausdorff space, then $M(X)$ denotes the space of regular complex Borel measures on X equipped with the total variation norm and the pairing $(g, \mu) = \int_X g(t)\,d\mu(t)$ identifies $M(X)$ with $C_0(X)^*$, the Banach space dual of $C_0(X)$. If $X = G$, a locally compact group, then $M(G)$

can be equipped with a convolution product. For $\mu, \nu \in M(G)$, there is a unique $\mu * \nu \in M(G)$ such that

$$\int_G \varphi(x)\, d(\mu * \nu)(x) = \int_G \int_G \varphi(xy)\, d\mu(x)\, d\nu(y),$$

for all $\varphi \in C_0(G)$, and we have $\|\mu * \nu\| \leq \|\mu\| \cdot \|\nu\|$. If δ_x denotes the point mass at $x \in G$, then $\int_G \varphi(y) d(\mu * \delta_x)(y) = \int_G R_x\varphi(y) d\mu(y)$ and $\int_G \varphi(y) d(\delta_x * \mu)(y) = \int_G L_{x^{-1}}\varphi(y) d\mu(y)$. Moreover, for $\mu \in M(G)$, define $\mu^* \in M(G)$ such that $\mu^*(E) = \overline{\mu(E^{-1})}$, for any Borel $E \subseteq G$. With this structure $M(G)$ is a Banach $*$-algebra with identity δ_e, called the *measure algebra* of G. For each $f \in L^1(G)$, there is a measure $\mu_f \in M(G)$ such that $d\mu_f(x) = f(x)\, dx$. This embeds $L^1(G)$ as a closed two-sided ideal in $M(G)$. Indeed, if $\nu \in M(G)$ and $f \in L^p(G)$, then the function

$$\nu * f : y \to \int_G f(x^{-1}y) d\nu(x)$$

belongs to $L^p(G)$ and, when $p = 1$, satisfies

$$\int_G \varphi(y) d(\nu * \mu_f)(y) = \int_G \varphi(y)(\nu * f)(y) dy,$$

for all $\varphi \in C_c(G)$. Similarly, $f * \nu$ is defined and $\mu_f * \nu = f * \nu$, for $f \in L^1(G)$ and $\nu \in M(G)$.

A closed subgroup H of G is locally compact with the relative topology. If H is an open subgroup of G, then H is automatically closed in G since $G \setminus H$ is the union of the open cosets xH, $x \notin H$. If H is a closed subgroup of G, then $\mu_G(H) > 0$ if and only if H is open in G. Thus, when H is open, $\mu_H(E) = \mu_G(E)$, for Borel subsets E of H, defines a Haar measure on H. However, if H is not open in G, then the relationship between the left Haar measure of H and that of G is not straightforward.

Let H be a closed subgroup of G and let $G/H = \{xH : x \in G\}$, the space of *left H-cosets*. Let $q : G \to G/H$, $q(x) = xH$, be the natural mapping. Then G/H is a locally compact Hausdorff space when given the quotient topology. Thus, q is an open as well as a continuous mapping.

If H is a normal closed subgroup of G, then G/H carries the structure of a group and hence is a locally compact group, the *quotient group* modulo H. The map q is then the *quotient homomorphism*.

If M is some topological group with identity element e_M and $\psi : G \to M$ is a continuous homomorphism, then $N = \{x \in G : \psi(x) = e_M\}$ is a closed normal subgroup of G. There is a unique injective homomorphism $\widetilde{\psi} : G/N \to M$

such that $\widetilde{\psi} \circ q = \psi$, and $\widetilde{\psi}$ is continuous since q is open. Suppose that ψ is surjective, M is locally compact, and G is σ-compact. Then $\widetilde{\psi}$ is a topological isomorphism. However, $\widetilde{\psi}$ need not be open when G is not σ-compact.

It is sometimes useful that every compact subset C of G is contained in an open σ-compact subgroup of G. In fact, choose a compact symmetric neighborhood V of e in G and let $A = C \cup C^{-1} \cup V$. Then $H = \cup_{n=1}^{\infty} A^n$ is an open σ-compact subgroup.

It is common for a locally compact group to be naturally acting on some other topological space. If Ω is a topological space and G is a locally compact group, a *left action* of G on Ω is a continuous map, $(x, \omega) \to x \cdot \omega$, of $G \times \Omega \to \Omega$ that satisfies $e \cdot \omega = \omega$ and $(xy) \cdot \omega = x \cdot (y \cdot \omega)$, for all $x, y \in G, \omega \in \Omega$. When we have such an action, for $\omega \in \Omega$, the *G-orbit of ω* is $G(\omega) = \{x \cdot \omega : x \in G\}$ and the *stabilizer of ω* is $G_\omega = \{x \in G : x \cdot \omega = \omega\}$. Since G_ω is clearly a subgroup of G, we often refer to such a G_ω as the *stability subgroup associated with ω*. If Ω is at least a T_1 space, then stability subgroups are closed. Note that $yG_\omega \to y \cdot \omega$ is a bijection of G/G_ω with $G(\omega)$. Moreover, for $y \in G$ and $\omega \in \Omega$, $G_{y \cdot \omega} = yG_\omega y^{-1}$, so stability subgroups associated with two points in the same G-orbit are conjugate in G.

1.2 Examples

The examples in this section serve the dual purpose of not only providing an idea of the nature of locally compact groups and their variety, but also introducing some of the particular groups or classes of groups that will be used later in the text.

Example 1.3 The set $\mathbb{R}$ of real numbers with addition and the usual topology has already been mentioned as a locally compact group. The integers $\mathbb{Z}$ form a closed subgroup of $\mathbb{R}$. Indeed, for every $a \in \mathbb{R}$, $\mathbb{Z}a = \{ka : k \in \mathbb{Z}\}$ is a closed subgroup of $\mathbb{R}$, and any proper closed subgroup of $\mathbb{R}$ is of this form.

Let $\mathbb{R}^* = \{a \in \mathbb{R} : a \neq 0\}$ and $\mathbb{R}^+ = \{a \in \mathbb{R} : a > 0\}$. Equipped with multiplication of real numbers, both $\mathbb{R}^*$ and $\mathbb{R}^+$ are locally compact groups with 1 as identity and $a^{-1} = 1/a$ as the inverse of a generic element a. Note that $\mathbb{R}^+$ is an open subgroup of $\mathbb{R}^*$.

If $\mathbb{C}$ is the field of complex numbers, then the *circle group*, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, is a compact group under multiplication. It is an exceptionally important group for harmonic analysis. If $a > 0$, then $\psi_a(t) = \exp(2\pi it/a)$, for $t \in \mathbb{R}$, defines a continuous homomorphism of $\mathbb{R}$ onto $\mathbb{T}$. Since $\mathbb{Z}a = \ker \psi_a$, $\widetilde{\psi}_a : t + \mathbb{Z}a \to \exp(2\pi it/a)$ identifies $\mathbb{R}/\mathbb{Z}a$ with $\mathbb{T}$.

The Haar measure $\mu_{\mathbb{R}}$ of $\mathbb{R}$ with $\mu_{\mathbb{R}}([0,1]) = 1$ is Lebesgue measure and the Haar measure on $\mathbb{T}$ is given by $\mu_{\mathbb{T}}(E) = \mu_{\mathbb{R}}(\psi_1^{-1}(E) \cap [0,1))$, for any Borel subset E of $\mathbb{T}$. That is, for any non-negative measurable function f on $\mathbb{T}$,

$$\int_{\mathbb{T}} f(z)dz = \int_{[0,1)} f(\exp(2\pi i t))dt,$$

where the integral on the right is a Lebesgue integral.

It is easiest to describe the Haar measure on $\mathbb{R}^*$ by showing the formula for invariant integration. If f is a non-negative measurable function on $\mathbb{R}^*$, then $\int_{\mathbb{R}^*} f(a) \frac{da}{|a|}$ satisfies

$$\int_{\mathbb{R}^*} f(ba) \frac{da}{|a|} = \int_{\mathbb{R}^*} f(a) \frac{da}{|a|},$$

for all $b \in \mathbb{R}^*$. Therefore, $\int_0^\infty f(a) \frac{da}{a}$ is integration with respect to the Haar measure on $\mathbb{R}^+$.

We will often specify a left Haar measure on a given locally compact group G by writing out the expression for integration with respect to the measure in a convenient parametrization of the elements of G.

Example 1.4 Let $n \in \mathbb{N}$ and $G_1, G_2, \ldots, G_n$ be locally compact groups. Then the Cartesian product, $G_1 \times G_2 \times \ldots \times G_n$, also denoted $\Pi_{j=1}^n G_j$, is a group when given the coordinatewise operations. With the product topology, $\prod_{j=1}^n G_j$ is a locally compact group called the *product group*. The left Haar measure on $\prod_{j=1}^n G_j$ is the product of the left Haar measures on the groups G_j. Thus, if f is a non-negative measurable function on $G = \prod_{j=1}^n G_j$,

$$\int_G f(x)dx = \int_{G_1} \int_{G_2} \ldots \int_{G_n} f(x_1, x_2, \ldots, x_n) dx_n \ldots dx_2 dx_1.$$

If $G_j = H$, a fixed locally compact group, for $j = 1, \ldots, n$, then $\prod_{j=1}^n G_j$ is denoted H^n. Thus, we have $\mathbb{R}^n$, $\mathbb{Z}^n$, $\mathbb{T}^n$, and combinations $\mathbb{R}^k \times \mathbb{Z}^l \times \mathbb{T}^m$, for any non-negative integers k, l, and m.

Example 1.5 We can equip $\mathbb{R} \times \mathbb{R}^+$ with a different multiplication. For $(b_1, a_1), (b_2, a_2) \in \mathbb{R} \times \mathbb{R}^+$ let $(b_1, a_1)(b_2, a_2) = (b_1 + a_1 b_2, a_1 a_2)$. Notice that $(0,1)(b,a) = (b,a)(0,1) = (b,a)$ and $(b,a)(-a^{-1}b, a^{-1}) = (-a^{-1}b, a^{-1})(b,a) = (0,1)$. Thus, this multiplication endows $\mathbb{R} \times \mathbb{R}^+$ with a group structure, and the resulting group will be denoted G_{aff}. The operations of multiplication and inversion are clearly continuous for the product topology. Thus G_{aff} is a locally compact group.

For $(b, a) \in G_{\mathrm{aff}}$, define a transformation of $\mathbb{R}$ by $(b, a) \cdot x = ax + b$, for all $x \in \mathbb{R}$. This is an affine transformation of $\mathbb{R}$, and every orientation-preserving affine transformation of $\mathbb{R}$ arises this way. Moreover, this action is consistent with the group product in G_{aff}. The group G_{aff} is called the *affine group* or, often, the *$ax + b$ group*.

We identify the left Haar measure on G_{aff} by providing a left-invariant integration formula. For a non-negative measurable function f on G_{aff},

$$\int_{G_{\mathrm{aff}}} f(z)dz = \int_{\mathbb{R}} \int_{\mathbb{R}^+} f(b, a)a^{-2} da\, db.$$

One can check that, for any $(b', a') \in G_{\mathrm{aff}}$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f((b', a')(b, a))a^{-2} da\, db = \int_{\mathbb{R}} \int_{\mathbb{R}} f(b, a)a^{-2} da\, db.$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}} f((b, a)(b', a'))a^{-2} da\, db &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(b + ab', aa')a^{-2} da\, db \\ &= a' \int_{\mathbb{R}} \int_{\mathbb{R}} f(b, a)a^{-2} da\, db.\end{aligned}$$

Thus, G_{aff} is nonunimodular and $\Delta_{G_{\mathrm{aff}}}(b, a) = a^{-1}$.

The previous example is a special case of a fruitful technique for constructing new locally compact groups from given ones. This is the semidirect product construction, which we present in detail because many of our examples dealt with later in the book arise in this manner. Let N and H be locally compact groups. Let $\mathrm{Aut}(N)$ denote the group of automorphisms of N. An automorphism of N is a topological group isomorphism of N with itself. Suppose that there is a homomorphism $\alpha : h \to \alpha_h$ of H into $\mathrm{Aut}(N)$ such that $(n, h) \to \alpha_h(n)$ is continuous from $N \times H$ to N. We use these data to form a locally compact group denoted $N \rtimes_\alpha H$, or simply $N \rtimes H$ if the homomorphism α is understood. As a set and topological space, $N \rtimes H = N \times H$. The group product of (n_1, h_1) with (n_2, h_2) in $N \rtimes H$ is given by

$$(n_1, h_1)(n_2, h_2) = (n_1 \alpha_{h_1}(n_2), h_1 h_2).$$

One checks easily that this product is associative, that (e_N, e_H) serves as the identity, where e_N and e_H are the identities of N and H, respectively, that $(n, h)^{-1} = (\alpha_{h^{-1}}(n^{-1}), h^{-1})$, and that the group operations are continuous. In short, $N \rtimes H$ is a locally compact group, called the *semidirect product* of N and H. To find the left Haar integral on $N \rtimes H$, one uses those on N and H together with a factor which records the amount by which α_h scales the left Haar integral of N. More precisely, fix $h \in H$ and define a measure μ_N^h on N by $\mu_N^h(E) = \mu_N(\alpha_h(E))$, for Borel subsets E of N. Since

$$\mu_N^h(nE) = \mu_N(\alpha_h(nE)) = \mu_N(\alpha_h(n)\alpha_h(E)) = \mu_N(\alpha_h(E)) = \mu_N^h(E),$$

μ_N^h is a left Haar measure on N. Thus, there exists $\delta(h) > 0$ so that $\mu_N^h(E) = \delta(h)\mu_N(E)$. Then $\delta : H \to \mathbb{R}^+$ is a continuous homomorphism and

$$\int_N f(x)dx = \delta(h) \int_N f(\alpha_h(x))dx,$$

for any non-negative measurable function f on N. Now the left Haar integral of any non-negative measurable function f on $N \rtimes H$ is given by

$$\int_{N \rtimes H} f(n,h)d(n,h) = \int_H \int_N f(n,h)\delta(h)^{-1}dndh.$$

The reader should check the left invariance. We will compute the modular function $\Delta_{N \rtimes H}$. Let $(m,k) \in N \rtimes H$. Then

$$\begin{aligned}
\int_{N \rtimes H} f((n,h)(m,k))d(n,h) &= \int_H \int_N f(n\alpha_h(m), hk)\delta(h)^{-1}dndh \\
&= \int_H \int_N f(\alpha_h(\alpha_{h^{-1}}(n)m), hk)\delta(h)^{-1}dndh \\
&= \int_H \int_N f(\alpha_h(nm), hk)dndh \\
&= \int_H \int_N f(\alpha_{hk^{-1}}(n), h)\Delta_N(m^{-1})\Delta_H(k^{-1})dndh \\
&= \int_H \int_N f(n,h)\delta(kh^{-1})\Delta_N(m^{-1})\Delta_H(k^{-1})dndh \\
&= \delta(k)(\Delta_N(m)\Delta_H(k))^{-1} \int_{N \rtimes H} f(n,h)d(n,h).
\end{aligned}$$

This shows that, for $(m,k) \in N \rtimes H$,

$$\Delta_{N \rtimes H}(m,k) = \Delta_N(m)\Delta_H(k)\delta(k)^{-1}.$$

If we define $\widetilde{N} = \{(n, e_H) : n \in N\}$ and $\widetilde{H} = \{(e_N, h) : h \in H\}$, then $\widetilde{N}$ and $\widetilde{H}$ are closed subgroups of $N \rtimes H$ that satisfy $\widetilde{N} \cap \widetilde{H} = \{e\}$ and $\widetilde{N}\widetilde{H} = N \rtimes H$, and $\widetilde{N}$ is normal in G. Moreover, for $h \in H$ and $n \in N$,

$$(e_H, h)(n, e_H)(e_H, h)^{-1} = (\alpha_h(n), e_H).$$

In general, if G is a locally compact group, we are sometimes able to find two closed subgroups N and H such that N is normal in G, $N \cap H = \{e\}$ and $NH = G$. In such a case we can view G as the semidirect product of N and H as follows. For $h \in H$, define α_h on N by $\alpha_h(n) = hnh^{-1}$, for all $n \in N$. Since N is normal, α_h is an automorphism of N and $h \to \alpha_h$ is a homomorphism of H into $\mathrm{Aut}(N)$. Also $(n, h) \to \alpha_h(n)$ is continuous from $N \times H$ to N. Thus, we can form the semidirect product $N \rtimes_\alpha H$, and the map $(n, h) \to nh$ is a topological isomorphism between $N \rtimes_\alpha H$ and G.

Example 1.6 Let $A \in M(n, \mathbb{R})$, the algebra of real $n \times n$-matrices. Then e^{tA} is defined, for all $t \in \mathbb{R}$, and $t \to e^{tA}$ is a continuous homomorphism of $\mathbb{R}$ into $GL(n, \mathbb{R})$, the group of nonsingular matrices in $M(n, \mathbb{R})$.

Consider $\mathbb{R}^n$ as an additive group and write the elements of $\mathbb{R}^n$ as column vectors. If we interpret e^{tA} as the automorphism $x \to e^{tA}x$ of $\mathbb{R}^n$, then we can form $\mathbb{R}^n \rtimes_A \mathbb{R}$. That is

$$\mathbb{R}^n \rtimes_A \mathbb{R} = \{(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

with the group product given by $(x, t)(y, s) = (x + e^{tA}y, t + s)$, for all $(x, t), (y, s) \in \mathbb{R}^n \rtimes_A \mathbb{R}$.

If $\delta = |\det A|$, then e^{tA} scales the Lebesgue (Haar) measure on $\mathbb{R}^n$ by a factor δ^t, for each $t \in \mathbb{R}$. Therefore, left Haar integration on $\mathbb{R}^n \rtimes_A \mathbb{R}$ is given by

$$\int_{\mathbb{R}^n \rtimes_A \mathbb{R}} f(x, t) d(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(x, t) \delta^{-t} dx dt.$$

Moreover, the modular function is given by $\Delta_{\mathbb{R}^n \rtimes_A \mathbb{R}}(x, t) = \delta^{-t}$, for all $(x, t) \in \mathbb{R}^n \rtimes_A \mathbb{R}$.

Recall Example 1.5. If $n = 1$ and $A = 1$, the map $\psi : \mathbb{R} \rtimes_A \mathbb{R} \to G_{\mathrm{aff}}$ given by $\psi(x, t) = (x, e^t)$, for $(x, t) \in \mathbb{R} \rtimes_A \mathbb{R}$, is an isomorphism of topological groups. In higher dimensions, the nature of $\mathbb{R}^n \rtimes_A \mathbb{R}$ and its representation theory varies greatly as the matrix A varies, providing a valuable family of examples.

Example 1.7 We already mentioned the general linear group $GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) : \det A \neq 0\}$. Since $M(n, \mathbb{R})$ can be identified with $\mathbb{R}^{n^2}$ via its entries as coordinates and $\det A$ is a polynomial in the entries of A, $GL(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$. Hence, it is locally compact in the topology inherited from $\mathbb{R}^{n^2}$. The entries of A^{-1} are rational functions of the entries of A with only factors of $\det A$ in the denominators. Thus $A \to A^{-1}$ is continuous, and multiplication of matrices is also easily seen to be continuous. So $GL(n, \mathbb{R})$ is a locally compact group.