

Index

A

absolutely monotone, 19
 absolutely pure jump path, 158
 abstract Wiener space, 309
 orthogonal invariance, 328
 ergodicity, 329
 adapted, 266
 σ -algebra
 atom in, 13
 tail, 2
 trivial, 2
 approximate identity, 16
 a.e. convergence of, 241
 Arcsine Law, 407
 a characterization of, 415
 for random variables, 409
 asymptotic, 32
 atom, 13
 Azema's Inequality, 264

B

Bachelier, 188
 barrier function, 423
 Beckner's inequality, 108
 Bernoulli multiplier, 101
 Bernoulli random variables, 5
 Bernstein polynomial, 17
 Berry–Esseen Theorem, 77
 Bessel operator, 350
 Beta function, 138
 Blumenthal's 0–1 Law, 426
 Bochner's Theorem, 119
 Borel measurable linear maps are continuous, 314
 Borel–Cantelli Lemma
 extended version of, 506
 martingale extension of, 229
 original version, 3
 Brownian motion, 177
 Erdős–Kac Theorem, 399
 Hölder continuity, 183
 in a Banach space, 359

iterated logarithm, 189, 366
 Lévy's martingale characterization, 282
 Lévy's modulus of continuity, 191
 non-differentiability, 183
 on a Banach space, 361
 pinned, 327, 334
 recurrence in one and two dimensions, 413
 reflection principle, 188, 294
 rotational invariance, 187
 scaling invariance, 187, 335
 for Banach space, 365
 strong law, 188
 time inversion, 187
 for Banach space, 365
 transience for $N \geq 3$, 414
 transition function for killed, 298
 variance of paths, 333
 with drift, 444
 Burkholder's Inequality, 262
 application to Fourier series, 263
 application to Walsh series, 264
 for continuous martingales, 289
 martingale comparison, 257
 for martingale square function, 262

C

Calderón–Zygmund Decomposition
 Gundy's for martingales, 227
 Cameron–Martin formula, 312
 Cameron–Martin space, 305
 classical, 305
 in general, 310
 capacity distribution, 499
 Chung's representation of, 500
 capacity potential, 497, 499
 capacity distribution, 499
 capacity, 499
 monotone continuity, 502
 capacity zero, 514
 Cauchy distribution, 149
 Cauchy initial value problem, 400
 centered Gaussian measure, 299
 non-degenerate, 306

centered random variable, 179
 Central Limit phenomenon, 60
 Central Limit Theorem
 basic case, 64
 Berry–Esseen, 77
 higher moments, 87
 Lindeberg, 61
 sub-Gaussian random variables, 89
 characteristic function, 82
 Chebyshev polynomial, 34
 Chebyshev’s inequality, 15
 Chernoff’s Inequality, 30
 Chung–Fuchs Theorem, 231
 conditional expectation, 194
 application to Fourier series, 204
 basic properties, 197
 existence and uniqueness, 195
 infinite measure, 200
 Banach space-valued case, 200
 Jensen’s Inequality for, 210
 properties, 197
 regular, 386
 versus orthogonal projection, 202
 conditional probability, 196
 as limit of naïve case, 209
 naïve case, 193
 regular version, 388
 conditional probability distribution, 388
 continuous martingale, 267
 Burkholder’s Inequality for, 289
 Doob–Meyer Theorem, 285
 exponential estimate, 291
 exponential martingale, 291
 continuous singular functions, 47
 convergence
 in law or distribution, 379
 weak, 116
 convolution, 63
 measure with measure, 115
 of function with measure, 83
 of functions, 63
 countably generated σ -algebra, 13
 covariance, 84
 Cramér’s Theorem, 27

D

DeFinetti, 219
 strong law, 220
 difference operator, 18
 Dirichlet problem, 418

balayage procedure, 426
 Courant–Friedrichs–Lewy scheme, 428
 finite difference scheme, 428
 Perron–Wiener solution, 423
 regular point, 421
 uniqueness, 463
 uniqueness criterion
 $N \geq 3$, 466
 $N \in \{1, 2\}$, 467
 distribution, 12
 function, 7
 Gaussian or normal, 85
 uniform, 6
 distribution of a stochastic process, 152
 Donsker’s Invariance Principle, 393
 Doob’s Decomposition, 213
 continuous case, *see* Doob–Meyer
 Doob’s Inequality
 Banach-valued case, 239
 continuous parameter, 270
 discrete parameter, 207
 Doob’s Stopping Time Theorem
 continuous parameter, 275
 discrete parameter, 213
 Doob–Meyer Decomposition, 285
 drift, 444
 Duhamel’s Formula, 282
 for Green function when $N = 2$, 482
 for Green function when $N \geq 3$, 476
 for killed Brownian motion, 298

E

eigenvalues for Dirichlet Laplacian, 450
 principal eigenvalue, 450
 Weyl’s asymptotic formula, 453
 empirical distribution, 384
 energy of a charge, 501
 equicontinuous family, 377
 Erdős–Kac Theorem, 399
 ergodic hypothesis
 continuous case, 254
 discrete case, 249
 ergodic theory
 Individual Ergodic Theorem
 continuous parameter, 254
 discrete parameter, 248
 stationary family, 251
 error function, 72
 Euler’s Gamma function, 32
 excessive function, 488

excessive function (*continued*)
 charge determined by, 494
 Riesz Decomposition of, 492

exchangeable random variables, 220
 Strong Law for, 220

exponential random variable, 161

extended stopping time, 278

F

Fernique's Theorem, 306
 application to functional analysis, 314

Feynman's representation, 303

Feynman–Kac
 formula, 403
 heat kernel, 437

fibering a measure, 389

first entrance time, asymptotics of distribution
 $N = 2$, 512
 $N \geq 3$, 509

first exit time, 419

fixed points of T_α , 92

Fourier transform, 82
 Beckner's inequality for, 108
 diagonalized by Hermite functions, 100
 for measure on Banach space, 301
 inversion formula, 98, 112
 of a function, 82
 of a measure, 82
 operator, 100
 Parseval's Identity for, 112

free fields
 Gaussian, 343
 ergodicity, 358
 existence of, 352

function
 characteristic, 82
 distribution, 7
 error, 72
 Euler's Beta, 138
 Euler's Gamma, 32
 excessive, 488
 Fourier transform of, 82
 Hermite, 100
 indicator, 4
 moment generating, 23
 logarithmic, 25
 normalized Hermite, 112
 probability generating, 19
 progressively measurable, 266

Rademacher, 5
 rapidly decreasing, 82
 tempered, 97

G

Gamma distribution, 138

Gamma function, 32

Gauss kernel, 23

Gaussian family, 179
 conditioning, 203

Gaussian measure
 on a Banach space, 299
 support of, 321

Gaussian random variable,
 independence vs. orthogonality, 94

generalized Poisson process, 171

Green function, 476
 for balls, 486
 Duhamel's Formula for $N = 2$, 482
 Duhamel's Formula for $N \geq 3$, 476
 properties when $N = 2$, 485

Green's Identity, 487

ground state, 439, 448
 associated eigenvalue, 439

ground state representation, 439

Guivarc'h recurrence lemma, 45, 256

H

Haar basis, 319

Hardy's Inequality, 238

Hardy–Littlewood Maximal Inequality, 235

harmonic function, 419
 Harnack's Inequality and Principle, 471
 Liouville Theorem, 472
 removable singularities for, 472

harmonic measure, 468
 for balls, 469
 for \mathbb{R}_+^N , 469

harmonic oscillator, 406

Harnack's Inequality, 471

Harnack's Principle, 471

heat equation, 400
 Cauchy initial value problem, 400

heat kernel, 429
 Dirichlet, 435
 Feynman–Kac, 437
 Hermite, 406, 454

heat transfer, Spitzer's asymptotic rate, 507

Hermite functions, 100
 eigenfunctions for Hermite operator, 454

Hermite functions (*continued*)
 Fourier eigenvectors, 100
 normalized, 112
 Hermite heat kernel, 406
 Hermite multiplier, 98
 Hermite operator, 406
 Hermite polynomials, 97
 L^p -estimate, 114
 Hewitt–Savage 0–1 Law, 221
 Hölder conjugate, 100
 hypercontractive, 105

I

independent
 events or sets, 1
 random variables, 4
 existence in general, 12
 existence of \mathbb{R} -valued sequences, 7
 σ -algebras, 1
 indicator function, 4
 inequality
 Azema's, 264
 Burkholder's, 262, 289
 Gross's logarithmic Sobolev, 114
 Harnack's, 471
 Jensen's, 210, 240
 Khinchine's, 94
 Kolmogorov's, 36
 Lévy's, 40
 Nelson's Hypercontractive, 106
 infinitely divisible, 115
 measure or law, 115
 inner product for measures, 230
 integer part, 5
 invariant set, 246

J

Jensen's Inequality, 210
 Banach-valued case, 240
 jump function, 156

K

Kac's Theorem, 252
 Kakutani's Theorem, 229
 kernel
 Gauss, 23
 Mehler's, 98
 Khinchine's Inequality, 94

Kolmogorov's
 continuity criterion, 182
 Extension or Consistency Theorem, 384
 Inequality, 36
 Strong Law, 38
 0–1 Law, 2
 Kronecker's Lemma, 37

L

λ -system, 8
 Laplace transform inversion formula, 21
 large deviations estimates, 28
 Law of Large Numbers
 Strong
 in Banach space, 241, 256, 384
 for empirical distribution, 384
 for exchangeable random variables, 220
 Kolmogorov's, 38
 Weak, 16
 refinement, 20, 44, 45
 Law of the Iterated Logarithm
 converse, 56
 proof of, 54
 statement, 49
 Strassen's Version, 340, 366
 Lebesgue's Differentiation Theorem, 237
 Lévy measure, 128
 Itô map for, 390
 Lévy operator, 268
 Lévy process, 152
 reflection, 292
 Lévy system, 134
 Lévy's Continuity Theorem, 118
 second version, 120
 Lévy–Cramér Theorem, 66
 Lévy–Khinchine formula, 136
 limit superior of sets, 2
 Lindeberg's Theorem, 61
 Lindeberg–Feller Theorem, 62
 Feller's part, 90
 Liouville Theorem, 472
 locally μ -integrable, 199
 Logarithmic Sobolev Inequality, 113
 for Bernoulli, 113
 logarithmic Sobolev Inequality
 for Gaussian, 114, 356
 lowering operator, 97

M

- marginal distribution, 83
- Markov property, 417
- martingale, 205
 - application to Fourier series, 263
 - continuous parameter, 267
 - complex, 267
 - Gundy's decomposition of, 227
 - Hahn decomposition of, 227
 - reversed, 217
 - Banach-valued case, 241
 - on σ -finite measure space, 233
- martingale convergence
 - continuous parameter, 271
 - Hilbert-valued case, 243
 - Marcinkewitz's Theorem, 207
 - preliminary version for Banach space, 239
 - second proof, 226
 - third proof, 227
 - via upcrossing inequality, 214
- maximal function
 - Hardy–Littlewood, 235
 - Hardy–Littlewood inequality, 236
- maximum principle of Phragmén–Lindelöf, 474
- Maxwell distribution for ideal gas, 70
- mean value
 - Banach space case, 199
 - vector-valued case, 84
- measure
 - invariant, 112
 - locally finite, 63
 - non-atomic, 381
 - product, 10
 - pushforward $\Phi_*\mu$ of μ under Φ , 12
- measure preserving, 244
- measures
 - consistent family, 383
 - tight, 376, 382
- median, 39
 - variational characterization, 43
- Mehler kernel, 98
- minimum principle, 130
 - strong, 405
 - weak, 404
- moment estimate for sums of independent random variables, 94
- moment generating function, 23
 - logarithmic, 25
- multiplier
 - Bernoulli, 101

Hermite, 98

N

- Nelson's Inequality, 106
- non-degenerate, 306
- non-negative definite function, 119
- non-negative linear functional, 374
- normal law, 23
 - fixed point characterization, 91
 - Lévy–Cramér Theorem, 66
 - standard, 23
- null set, *see* \mathbb{P} -null set

O

- operator
 - Fourier, 100
 - hypercontractive, 105
 - lowering, 97
 - raising, 96
- optional stopping time, 280
- Ornstein–Uhlenbeck process, 344
 - ancient, 345
 - associated martingales, 415
 - Gaussian description, 344
 - Hermite heat kernel, 454
 - reversible, 346
 - in Banach space, 365

P

- Paley–Littlewood Inequality for Walsh series, 264
- Paley–Wiener map, 312
 - as a stochastic integral, 316
- Parseval's Identity, 112
- path properties, 158
 - absolutely pure jump, 158
 - piecewise constant, 158
- Phragmén–Lindelöf, 474
- pinned Brownian motion, 327
- π -system, 8
- \mathbb{P} -null set, 194
- Poincaré's Inequality for Gaussian, 355
- Poisson jump process, 168
 - Itô's construction of, 390
- Poisson kernel, 149
 - for upper half-space, 429
 - for ball via Green's Identity, 487
- Poisson measure, 122
 - generalized, 171
 - simple, 161

Poisson point process, 176
 Poisson problem, 475
 Poisson process, 161, 163
 associated with π_M , 164
 generalized, 171
 jump distribution, 163
 rate, 163
 simple, 161
 Poisson random variable, N -valued, 21
 Poisson's formula, 469
 Polish space, 367
 potential, 487
 charge determined by, 494
 in terms of excessive functions, 494
 principle of accompanying laws, 380
 probability space, 1
 process
 Brownian motion, 177
 with drift, 444
 Ornstein–Uhlenbeck, 344
 stationary, 345
 process with independent, homogeneous
 increments, 152
 product measure, 10
 progressively measurable, 205, 266
 versus adapted, 267
 pushforward measure $\Phi_*\mu$, 12

Q

quitting time, 500

R

Rademacher functions, 5
 Radon–Nikodym derivatives, martingale
 interpretation, 216
 raising operator, 96
 random variable, 4
 N -valued Poisson, 21
 Bernoulli, 5
 characteristic function, 82
 convergence in law, 379
 Gaussian or normal, 23
 vector-valued case, 85
 median of, 39
 sub-Gaussian, 88
 symmetric, 44
 uniformly integrable, 15
 variance of, 15
 rapidly decreasing, 9, 82
 Rayleigh's Random Flights Model, 396, 399

recurrence of Brownian motion, 413
 reflection principle
 Brownian motion, 188, 294
 for independent random variables, 40
 regular point, 421, 427
 exterior cone condition, 427
 probabilistic criterion, 421
 Wiener's test for, 504
 removable singularity, 472
 return time, Kac's Theorem for, 252
 Riemann–Lebesgue Lemma, 121
 Riesz Decomposition Theorem, 492
 Robin's constant, 485

S

semigroup, hypercontractive estimate, 105
 shift invariant, 251
 σ -algebra, countably generated, 13
 simple Poisson process, 163
 run at rate α , 163
 Sobolev space, 350
 square function, Burkholder's Inequality
 for, 262
 stable laws, 141
 $\frac{1}{2}$ order one-sided
 Brownian motion, 281
 density, 149
 characterization, 144
 one-sided, 147
 density, 148
 symmetric, 146
 densities, 149
 state space, 152
 stationary, 251
 stationary family
 canonical setting for, 251
 Kac's Theorem for, 252
 stationary process, 345
 statistical mechanics, derivation of Maxwell
 distribution, 70
 Stein's method, 72
 Stirling's formula, 32, 70
 stochastic integral, 316
 stochastic process, 152
 adapted, 266
 continuous, 266
 distribution of, 152
 independent increments, 152
 modification, 189
 reversible, 346

- stochastic process (*continued*)
 right-continuous, 266
 state of, 152
 stochastic continuity, 189
- stopping time, 212
 continuous parameter, 272
 discrete case, 212
 extended, 278
 old definition, 280
 optional, 280
- Stopping Time Theorem
 Doob's, continuous parameter, 275
 Doob's, discrete parameter, 213
 Hunt's, continuous parameter, 275
 Hunt's, discrete parameter, 213
- Strassen's Theorem, 340
 Brownian formulation of, 363
- Strong Law of Large Numbers, 23
 for Brownian motion, 188
 for empirical distribution, 384
 in Banach space, 241, 256, 384
 Kolmogorov's, 38
- strong Markov property, 417
- Strong Minimum Principle, 405
- strong topology on $\mathbf{M}_1(E)$, 369
 not metrizable, 381
- sub-Gaussian random variables, moment estimates, 93
- submartingale, 205
 continuous parameter, 267
 Doob's Decomposition, 213
 Doob's Inequality
 continuous parameter, 270
 discrete parameter, 206
 Doob's Upcrossing Inequality, 214
 reversed, 217
 σ -finite measure space, 233
 stopping time theorem
 Doob's
 discrete parameter, 212
 Doob's continuous parameter, 275
 Hunt's
 discrete parameter, 213
 Hunt's continuous parameter, 275
- subordination, 148
- symmetric difference of sets, 246
- symmetric random variable, 44
 moment relations, 45
- T**
- tail σ -algebra, 2
 and exchangability, 220
 ergodicity of, 256
- tempered, 97
- tempered distribution, 350
- tight, 376, 382
 for finite measures, 382
- time reversal, 335
- time-shift map, 416
- Tonelli's Theorem, 4
- transform
 Fourier, *see* Fourier transform
 Laplace, 21
 Legendre, 26
- transformation, measure preserving, 244
- transient, 414
- transition probability, 112
- U**
- uniform norm $\| \cdot \|_u$, 17
- uniform topology on $\mathbf{M}_1(E)$, 367
- uniformly distributed, 6
- uniformly integrable, 15
- unit exponential random variable, 161
- V**
- variance, 15
- variation norm, 368
- W**
- Walsh functions, 264
- weak convergence, 116
 equivalent formulations, 372
 principle of accompanying laws, 380
- Weak Law of Large Numbers, 16
- Weak Minimum Principle, 404
- weak topology on $\mathbf{M}_1(E)$, 370
 completeness, 377
 Prohorov metric for, 379
 separable, 376, 382
- weak-type inequality, 207
- Weierstrass's Approximation Theorem, 17
- Wiener measure, 301
 Arcsine law, 407
 Feynman's representation, 303
 Markov property, 417
 translation by \mathbf{x} , 401
- Wiener series, 318
 classical case, 334
- Wiener's test for regularity, 504