

Essential Mathematical Methods for the Physical Sciences

The mathematical methods that physical scientists need for solving substantial problems in their fields of study are set out clearly and simply in this tutorial-style textbook. Students will develop problem-solving skills through hundreds of worked examples, self-test questions and homework problems. Each chapter concludes with a summary of the main procedures and results and all assumed prior knowledge is summarized in one of the appendices. Over 300 worked examples show how to use the techniques and around 100 self-test questions in the footnotes act as checkpoints to build student confidence. Nearly 400 end-of-chapter problems combine ideas from the chapter to reinforce the concepts. Hints and outline answers to the odd-numbered problems are given at the end of each chapter, with fully worked solutions to these problems given in the accompanying *Student Solution Manual*. Fully worked solutions to all problems, password-protected for instructors, are available at www.cambridge.org/essential.

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Contents

<i>Preface</i>	<i>page</i> xiii
<i>Review of background topics</i>	xvi

1	Matrices and vector spaces	1
1.1	Vector spaces	2
1.2	Linear operators	5
1.3	Matrices	7
1.4	Basic matrix algebra	8
1.5	Functions of matrices	13
1.6	The transpose of a matrix	13
1.7	The complex and Hermitian conjugates of a matrix	14
1.8	The trace of a matrix	16
1.9	The determinant of a matrix	17
1.10	The inverse of a matrix	21
1.11	The rank of a matrix	25
1.12	Simultaneous linear equations	27
1.13	Special types of square matrix	36
1.14	Eigenvectors and eigenvalues	40
1.15	Determination of eigenvalues and eigenvectors	45
1.16	Change of basis and similarity transformations	49
1.17	Diagonalization of matrices	51
1.18	Quadratic and Hermitian forms	53
1.19	Normal modes	58
1.20	The summation convention	67
	Summary	68
	Problems	72
	Hints and answers	83
2	Vector calculus	87
2.1	Differentiation of vectors	87
2.2	Integration of vectors	92
2.3	Vector functions of several arguments	93
2.4	Surfaces	94
2.5	Scalar and vector fields	96

2.6	Vector operators	96
2.7	Vector operator formulae	103
2.8	Cylindrical and spherical polar coordinates	107
2.9	General curvilinear coordinates	113
	Summary	119
	Problems	121
	Hints and answers	126
3	Line, surface and volume integrals	128
3.1	Line integrals	128
3.2	Connectivity of regions	134
3.3	Green's theorem in a plane	135
3.4	Conservative fields and potentials	138
3.5	Surface integrals	141
3.6	Volume integrals	147
3.7	Integral forms for grad, div and curl	149
3.8	Divergence theorem and related theorems	153
3.9	Stokes' theorem and related theorems	158
	Summary	161
	Problems	163
	Hints and answers	168
4	Fourier series	170
4.1	The Dirichlet conditions	170
4.2	The Fourier coefficients	172
4.3	Symmetry considerations	174
4.4	Discontinuous functions	175
4.5	Non-periodic functions	176
4.6	Integration and differentiation	179
4.7	Complex Fourier series	180
4.8	Parseval's theorem	181
	Summary	183
	Problems	185
	Hints and answers	189
5	Integral transforms	191
5.1	Fourier transforms	191
5.2	The Dirac δ -function	197
5.3	Properties of Fourier transforms	202
5.4	Laplace transforms	210
5.5	Concluding remarks	217
	Summary	218

Problems	219
Hints and answers	226
6 Higher-order ordinary differential equations	228
6.1 General considerations	229
6.2 Linear equations with constant coefficients	233
6.3 Linear recurrence relations	237
6.4 Laplace transform method	242
6.5 Linear equations with variable coefficients	244
6.6 General ordinary differential equations	258
Summary	262
Problems	264
Hints and answers	271
7 Series solutions of ordinary differential equations	273
7.1 Second-order linear ordinary differential equations	273
7.2 Ordinary and singular points of an ODE	275
7.3 Series solutions about an ordinary point	277
7.4 Series solutions about a regular singular point	280
7.5 Obtaining a second solution	286
7.6 Polynomial solutions	290
Summary	292
Problems	293
Hints and answers	297
8 Eigenfunction methods for differential equations	298
8.1 Sets of functions	300
8.2 Adjoint, self-adjoint and Hermitian operators	303
8.3 Properties of Hermitian operators	305
8.4 Sturm–Liouville equations	308
8.5 Superposition of eigenfunctions: Green’s functions	312
Summary	315
Problems	316
Hints and answers	320
9 Special functions	322
9.1 Legendre functions	322
9.2 Associated Legendre functions	333
9.3 Spherical harmonics	339
9.4 Chebyshev functions	341

9.5	Bessel functions	347
9.6	Spherical Bessel functions	360
9.7	Laguerre functions	361
9.8	Associated Laguerre functions	366
9.9	Hermite functions	369
9.10	The gamma function and related functions	373
	Summary	377
	Problems	380
	Hints and answers	385
10	Partial differential equations	387
10.1	Important partial differential equations	387
10.2	General form of solution	392
10.3	General and particular solutions	393
10.4	The wave equation	405
10.5	The diffusion equation	408
10.6	Boundary conditions and the uniqueness of solutions	411
	Summary	413
	Problems	414
	Hints and answers	419
11	Solution methods for PDEs	421
11.1	Separation of variables: the general method	421
11.2	Superposition of separated solutions	425
11.3	Separation of variables in polar coordinates	433
11.4	Integral transform methods	455
11.5	Inhomogeneous problems – Green's functions	460
	Summary	476
	Problems	479
	Hints and answers	486
12	Calculus of variations	488
12.1	The Euler–Lagrange equation	489
12.2	Special cases	490
12.3	Some extensions	494
12.4	Constrained variation	496
12.5	Physical variational principles	498
12.6	General eigenvalue problems	501
12.7	Estimation of eigenvalues and eigenfunctions	503
12.8	Adjustment of parameters	506
	Summary	507

Problems	509
Hints and answers	514
13 Integral equations	516
13.1 Obtaining an integral equation from a differential equation	516
13.2 Types of integral equation	517
13.3 Operator notation and the existence of solutions	518
13.4 Closed-form solutions	519
13.5 Neumann series	526
13.6 Fredholm theory	528
13.7 Schmidt–Hilbert theory	529
Summary	532
Problems	534
Hints and answers	538
14 Complex variables	540
14.1 Functions of a complex variable	541
14.2 The Cauchy–Riemann relations	543
14.3 Power series in a complex variable	547
14.4 Some elementary functions	549
14.5 Multivalued functions and branch cuts	551
14.6 Singularities and zeros of complex functions	553
14.7 Conformal transformations	556
14.8 Complex integrals	559
14.9 Cauchy’s theorem	563
14.10 Cauchy’s integral formula	566
14.11 Taylor and Laurent series	568
14.12 Residue theorem	573
Summary	576
Problems	578
Hints and answers	580
15 Applications of complex variables	582
15.1 Complex potentials	582
15.2 Applications of conformal transformations	587
15.3 Definite integrals using contour integration	590
15.4 Summation of series	597
15.5 Inverse Laplace transform	599
15.6 Some more advanced applications	602
Summary	605

Contents

Problems	606
Hints and answers	610
16 Probability	612
16.1 Venn diagrams	612
16.2 Probability	617
16.3 Permutations and combinations	627
16.4 Random variables and distributions	633
16.5 Properties of distributions	638
16.6 Functions of random variables	642
16.7 Generating functions	646
16.8 Important discrete distributions	654
16.9 Important continuous distributions	666
16.10 The central limit theorem	681
16.11 Joint distributions	683
16.12 Properties of joint distributions	685
Summary	691
Problems	695
Hints and answers	703
17 Statistics	705
17.1 Experiments, samples and populations	705
17.2 Sample statistics	706
17.3 Estimators and sampling distributions	713
17.4 Some basic estimators	721
17.5 Data modeling	730
17.6 Hypothesis testing	735
Summary	755
Problems	759
Hints and answers	764
A Review of background topics	766
A.1 Arithmetic and geometry	766
A.2 Preliminary algebra	768
A.3 Differential calculus	770
A.4 Integral calculus	771
A.5 Complex numbers and hyperbolic functions	773
A.6 Series and limits	774
A.7 Partial differentiation	777
A.8 Multiple integrals	778
A.9 Vector algebra	779
A.10 First-order ordinary differential equations	781

B	Inner products	782
C	Inequalities in linear vector spaces	784
D	Summation convention	786
E	The Kronecker delta and Levi–Civita symbols	789
F	Gram–Schmidt orthogonalization	793
G	Linear least squares	795
H	Footnote answers	797
	<i>Index</i>	810

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Frontmatter
[More information](#)

Preface

Since *Mathematical Methods for Physics and Engineering* (Cambridge: Cambridge University Press, 1998) by Riley, Hobson and Bence, hereafter denoted by *MMPE*, was first published, the range of material it covers has increased with each subsequent edition (2002 and 2006). Most of the additions have been in the form of introductory material covering polynomial equations, partial fractions, binomial expansions, coordinate geometry and a variety of basic methods of proof, though the third edition of *MMPE* also extended the range, but not the general level, of the areas to which the methods developed in the book could be applied. Recent feedback suggests that still further adjustments would be beneficial. In so far as content is concerned, the inclusion of some additional introductory material such as powers, logarithms, the sinusoidal and exponential functions, inequalities and the handling of physical dimensions, would make the starting level of the book better match that of some of its readers.

To incorporate these changes, and others to increase the user-friendliness of the text, into the current third edition of *MMPE* would inevitably produce a text that would be too ponderous for many students, to say nothing of the problems the physical production and transportation of such a large volume would entail. It is also the case that for students for whom a course on mathematical methods is *not* their first engagement with mathematics beyond high school level, all of the additional introductory material, as well as some of that presented in the early chapters of the original *MMPE*, would be ground already covered. For such students, typically those who have already taken two or three semesters of calculus, and perhaps an introductory course in ordinary differential equations, much of the first half of such an omnibus edition would be redundant.

For these reasons, we present under the current title, *Essential Mathematical Methods for the Physical Sciences*, an alternative edition of *MMPE*, one that focuses on the core of a putative extended third edition, omitting, except in summary form, all of the “mathematical tools” at one end, and some of the more specialized topics that appear in the third edition at the other. The emphasis is very much on developing the *methods* required by physical scientists before they can apply their knowledge of mathematical concepts to significant problems in their chosen fields.

For the record, we note that the more advanced topics in the third edition of *MMPE* that have fallen victim to this approach are quantum operators, tensors, group and representation theory, and numerical methods. The chapters on special functions, and the applications of complex variables have both been reduced to some extent, as have those on probability and statistics.

At the other end of the spectrum, the excised introductory material has not been altogether lost. Indeed, Appendix A of the present text consists entirely of summaries, in the style described in the penultimate paragraph of this Preface, of the material that

Preface

is presumed to have been previously studied and mastered by the student. Clearly it can be used both as a reference/reminder and as an indicator of some missing background knowledge.

One aspect that has remained constant throughout the three editions of *MMPE*, is the general style of presentation of a topic – a qualitative introduction, physically based wherever possible, followed by a more formal presentation or proof, and finished with one or two full-worked examples. This format has been well received by reviewers, and there is no reason to depart from its basic structure.

In terms of style, many physical science students appear to be more comfortable with presentations that contain significant amounts of verbal explanation and comment, rather than with a series of mathematical equations the last line of which implies “job done”. We have made changes that move the text in this direction. As is explained below, we also feel that if some of the advantages of small-group face-to-face teaching could be reflected in the written text, many students would find it beneficial.

One of the advantages of an oral approach to teaching, apparent to some extent in the lecture situation, and certainly in what are usually known as tutorials,¹ is the opportunity to follow the exposition of any particular point with an immediate short, but probing, question that helps to establish whether or not the student has grasped that point. This facility is not normally available when instruction is through a written medium, without having available at least the equipment necessary to access the contents of a storage disc.

In this book we have tried to go some way towards remedying this by making a non-standard use of footnotes. Some footnotes are used in traditional ways, to add a comment or a pertinent but not essential piece of additional information, to clarify a point by restating it in slightly different terms, or to make reference to another part of the text or an external source. However, about half of the nearly 300 footnotes in *this* book contain a question for the reader to answer or an instruction for them to follow; neither will call for a lengthy response, but in both cases an understanding of the associated material in the text will be required. This parallels the sort of follow-up a student might have to supply orally in a small-group tutorial, after a particular aspect of their written work has been discussed.

Naturally, students should attempt to respond to footnote questions using the skills and knowledge they have acquired, re-reading the relevant text if necessary, but if they are unsure of their answer, or wish to feel the satisfaction of having their correct response confirmed, they can consult the specimen answers given in Appendix H. Equally, footnotes in the form of observations will have served their purpose when students are consistently able to say to themselves “I didn’t need that comment – I had already spotted and checked that particular point”.

One further feature of the present volume is the inclusion at the end of each chapter, just before the problems begin, of a summary of the main results of that chapter. For some areas, this takes the form of a tabulation of the various case types that may arise in the context of the chapter; this should help the student to see the parallels between situations which in the main text are presented as a consecutive series of often quite lengthy pieces of mathematical development. It should be said that in such a summary it is not possible to state every detailed condition attached to each result, and the reader should consider

.....
¹ But in Cambridge are called “supervisions”!

Preface

the summaries as reminders and formulae providers, rather than as teaching text; that is the job of the main text and its footnotes.

Finally, we note, for the record, that the format and number of problems associated with the various remaining chapters have not been changed significantly, though problems based on excised topics have naturally been omitted. This means that hints or abbreviated solutions to all 200 odd-numbered problems appear in this text, and fully worked solutions of the same problems can be found in an accompanying volume, the *Student Solution Manual for Essential Mathematical Methods for the Physical Sciences*. Fully worked solutions to all problems, both odd- and even-numbered, are available to accredited instructors on the password-protected website www.cambridge.org/essential. Instructors wishing to have access to the website should contact solutions@cambridge.org for registration details.

Review of background topics

As explained in the Preface, this book is intended for those students who are pursuing a course in mathematical methods, but for whom it is not their first engagement with mathematics beyond high school level. Typically, such students will have already taken two or three semesters of calculus, and perhaps an introductory course in ordinary differential equations. The emphasis in the text is very much on developing the *methods* required by physical scientists before they can apply their knowledge of mathematical concepts to significant problems in their chosen fields; the basic mathematical “tools” that the student is presumed to have mastered are therefore not discussed in any detail.

However this introductory note and the associated appendix (Appendix A) are included both to act as a reference (or reminder) and to be an indicator of any presumed, but missing, topics in the student’s background knowledge. The appendix consists of summary pages for ten major topic areas, ranging from powers and logarithms at one extreme to first-order ordinary differential equations at the other. The style they adopt is identical to that used for the chapter summary pages in the 17 main chapters of the book. It should be noted that in such summaries it is not possible to state every detailed condition attached to each result. In the areas covered in Appendix A, there are very few subtle situations to consider, but the reader should be aware that they may exist.

Naturally, being only summaries, the various sections of the appendix will not be sufficient for the student who needs to catch up in some area, to learn the particular topics from scratch. A more elementary text will clearly be needed; *Foundation Mathematics for the Physical Sciences* written by the current authors would be one such possibility.