1

t-algebras and differentials

From now on k denotes a fixed ground field. Whenever we consider a k-algebra or a bimodule, we always assume that the field k acts centrally on them. We start with some basic notions and notation, and some elementary remarks.

Definition 1.1. *We say that the k-algebra T is freely generated by the pair (A, V) if A is a k-subalgebra of T, V is a A-A-subbimodule of T, and the following universal property is satisfied: for any k-algebra B, any morphism of k-algebras $A \xrightarrow{\phi_0} B$ and any morphism of A-A-bimodules $V \xrightarrow{\phi_1} B$, where the structure of A-A-bimodule of B is obtained by restriction through ϕ_0, there exists a unique morphism of k-algebras $T \xrightarrow{\phi} B$, which extends both ϕ_0 and ϕ_1.*

Definition 1.2. *Consider a k-algebra A and any A-A-bimodule V. For $i \geq 2$, we write $V^{\otimes i}$ for the tensor product $V \otimes_A V \otimes_A \cdots \otimes_A V$ of i copies of V, and set $V^{\otimes 0} = A$ and $V^{\otimes 1} = V$. The vector space $T_A(V) = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ admits a natural structure of k-algebra with product determined by the canonical isomorphism $V^{\otimes i} \otimes_A V^{\otimes j} \longrightarrow V^{\otimes(i+j)}$. The algebra $T_A(V)$ is called the tensor algebra of V over A.*

Lemma 1.3. *Consider a k-algebra A and any A-A-bimodule V. Then:*

(1) The tensor algebra $T_A(V)$ is freely generated by (A, V).
(2) If the algebra T is freely generated by (A, V), the morphism $T_A(V) \longrightarrow T$ determined by the inclusions of A and V in T is an isomorphism.
(3) Assume that T is an algebra freely generated by (A, V), and that $V = V' \bigoplus V''$ is a bimodule decomposition of the A-A-bimodule V. Then, the subalgebra A' of T generated by $A \cup V'$ is freely generated by (A, V') and T is freely generated by $(A', A'V''A')$.

Proof. (1) and (2) are easy to show. We show (3): let $\pi_1 : V \longrightarrow V'$ be the projection and consider the algebra morphism $T \xrightarrow{\pi} T_A(V')$ determined by the inclusion of A in $T_A(V')$ and π_1. Consider also the morphism $T_A(V') \xrightarrow{\sigma} T$ determined by the inclusion of A and V' in T. Then, clearly, $\pi\sigma$ is the identity map, and the restriction of σ to its image provides the isomorphism $T_A(V') \cong A'$. Now, consider the morphism $T_{A'}(A'V''A') \xrightarrow{\phi} T$ determined by the inclusions of A' and $A'V''A'$ in T. Consider also the morphism of A-A-bimodules $V \xrightarrow{\psi_1} T_{A'}(A'V''A')$, which maps each $v' \in V'$ onto $v' \in A'$, and each $v'' \in V''$ onto $v'' \in A'V''A'$. Then, the morphism $T \xrightarrow{\psi} T_{A'}(A'V''A')$ determined by the inclusion of A in A' and ψ_1 is an inverse for ϕ. □

Definition 1.4. *We say that T is a graded k-algebra if T is a k-algebra which admits a vector space decomposition $T = \oplus_{i\geq 0}[T]_i$ such that $[T]_i[T]_j \subseteq [T]_{i+j}$, for all i, j. Thus, $[T]_0$ is a subalgebra of T and each $[T]_i$ is a $[T]_0$-subbimodule of T. The elements $a \in [T]_i$ are called homogeneous of degree i, and we write* $\deg(a) = i$.

We say that T is a t-algebra if T is a graded k-algebra and T is freely generated by the pair $([T]_0, [T]_1)$.

Remark 1.5. *Consider a k-algebra A and any A-A-bimodule V:*

(1) The tensor algebra $T_A(V)$ with its standard grading given by $[T_A(V)]_i = V^{\otimes i}$ is a graded algebra.
(2) If T is a t-algebra, and we make $A = [T]_0$ and $V = [T]_1$, then there is an isomorphism of graded k-algebras $T_A(V) \cong T$ if we consider the standard grading on $T_A(V)$. In particular, for each n, the product of n elements induces an isomorphism

$$[T]_1 \otimes_{[T]_0} [T]_1 \otimes_{[T]_0} \cdots \otimes_{[T]_0} [T]_1 \xrightarrow{\cong} [T]_n.$$

We often identify both bimodules.

Definition 1.6. *Assume T is a graded k-algebra. Then we say that δ is a differential on T if $\delta : T \to T$ is a linear transformation such that $\delta([T]_i) \subseteq [T]_{i+1}$, for all i, and δ satisfies Leibniz rule: $\delta(ab) = \delta(a)b + (-1)^{\deg(a)}a\delta(b)$, for all homogeneous elements $a, b \in T$.*

Remark 1.7. *If T is a graded k-algebra and δ is a differential on T, then:*

(1) By induction, we obtain the following formula, for any homogeneous elements $t_1, t_2, \ldots, t_n \in T$

$$\delta(t_1 t_2 \cdots t_n) = \sum_{i=1}^{n} (-1)^{(\sum_{j=1}^{i-1} \deg(t_j))} t_1 t_2 \cdots t_{i-1} \delta(t_i) t_{i+1} t_{i+2} \cdots t_n.$$

(2) *The linear map* $\delta^2 : T \longrightarrow T$ *satisfies* $\delta^2(ab) = \delta^2(a)b + a\delta^2(b)$, *for any homogeneous elements* $a, b \in T$. *Again, by induction, we obtain the following formula, for any homogeneous elements* $t_1, \ldots, t_n \in T$

$$\delta^2(t_1t_2\cdots t_n) = \sum_{i=1}^{n} t_1t_2\cdots t_{i-1}\delta^2(t_i)t_{i+1}t_{i+2}\cdots t_n.$$

(3) *From (1), (2) and (1.5), we obtain that if* T *is a t-algebra, the differential* δ *and its square* δ^2 *are determined by their values on* $A = [T]_0$ *and on* $V = [T]_1$. *In particular, we can also derive that* $\delta^2(A) = 0$ *and* $\delta^2(V) = 0$ *imply* $\delta^2 = 0$.

Lemma 1.8. *Let* T *be a t-algebra. Denote* $A = [T]_0$ *and* $V = [T]_1$. *Assume we have a pair of linear maps* $\delta_0 : A \longrightarrow [T]_1$ *and* $\delta_1 : V \longrightarrow [T]_2$ *such that* $\delta_0(ab) = \delta_0(a)b + a\delta_0(b)$, $\delta_1(av) = \delta_0(a)v + a\delta_1(v)$ *and* $\delta_1(va) = \delta_1(v)a - v\delta_0(a)$, *for* $a, b \in A$ *and* $v \in V$. *Then, these maps extend uniquely to a differential* $\delta : T \longrightarrow T$.

Proof. Since T is a t-algebra, freely generated by (A, V), we may assume that $T = T_A(V)$, with its standard grading. We shall define a linear map δ_n from each of the direct summands $V^{\otimes n}$ of T to T. We use the same symbols δ_0 and δ_1 to denote their compositions with the inclusions to T. Then, for $n \geq 2$, define δ_n by the formula

$$\delta_n(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n}(-1)^{(i-1)}v_1v_2\cdots v_{i-1}\delta_1(v_i)v_{i+1}v_{i+2}\cdots v_n,$$

where each $v_i \in V$. See Remark (1.7). This formula yields a well-defined linear map $\delta_n : V^{\otimes n} \longrightarrow T$, because $\delta_1(av_i) = \delta_0(a)v_i + a\delta_1(v_i)$ and $\delta_1(v_{i-1}a) = \delta_1(v_{i-1})a - v_{i-1}\delta_0(a)$, for any $a \in A$. Then, there is a linear map $\delta : T \longrightarrow T$ which extends all these maps δ_n. It is clear that $\delta([T]_i) \subseteq [T]_{i+1}$. It remains to show that δ satisfies Leibniz rule. By assumption, δ already satisfies Leibniz rule for products of the form ab, av and va, with $a, b \in A$ and $v \in V$. From this and the definition of δ, it follows that δ satisfies Leibniz rule for products of the form aw and wa, with $a \in A$ and $w \in V^{\otimes n}$. To finish our proof, it is enough to show that given $u_n = \otimes_{s=1}^{n} v_s$ and $w_m = \otimes_{r=1}^{m} v'_r$, with $v_s, v'_r \in V$, then $\delta(u_n \otimes w_m) = \delta(u_n) \otimes w_m + (-1)^{\deg(u_n)}u_n \otimes \delta(w_m)$. This is a straightforward calculation using the definition of δ. □

2

Ditalgebras and modules

In this section, we introduce the basic objects studied in these notes. Namely, ditalgebras and their categories of modules. Its study constitutes a natural generalization of the theory of algebras and their categories of modules. At the same time, it has proved to be a useful tool in establishing some deep results in representation theory of algebras.

Definition 2.1. *A differential t-algebra or ditalgebra $\mathcal{A}$ is by definition a pair $\mathcal{A} = (T, \delta)$, where T is a t-algebra and δ is a differential on T satisfying $\delta^2 = 0$.*

A morphism of ditalgebras $\phi : (T, \delta) \to (T', \delta')$ is a morphism of k-algebras $\phi : T \to T'$, satisfying $\phi([T]_i) \subseteq [T']_i$, for all i, and $\delta'\phi = \phi\delta$.

Clearly, we can consider the category of ditalgebras over k, where the morphisms are composed as maps.

Definition 2.2. *The category of modules (or representations) of the ditalgebra $\mathcal{A} = (T, \delta)$, denoted by $\mathcal{A}$-Mod, is defined as follows. Denote by $A = A_{\mathcal{A}} = [T]_0$, a k-subalgebra of T, and by $V = V_{\mathcal{A}} = [T]_1$, an A-A-subbimodule of T. The objects of $\mathcal{A}$-Mod are all the A-modules. Given $M, N \in \mathcal{A}$-Mod, a morphism $f : M \to N$ in $\mathcal{A}$-Mod is a pair $f = (f^0, f^1)$, with $f^0 \in \mathrm{Hom}_k(M, N)$ and $f^1 \in \mathrm{Hom}_{A\text{-}A}(V, \mathrm{Hom}_k(M, N))$ satisfying that*

$$af^0(m) = f^0(am) + f^1(\delta(a))(m),$$

for any $a \in A$ and $m \in M$. The Hom-space in this category is denoted by $\mathrm{Hom}_{\mathcal{A}}(M, N)$. Given $f \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(N, L)$ in $\mathcal{A}$-Mod, consider the composition of morphisms of A-A-bimodules

$$V \otimes_A V \xrightarrow{g^1 \otimes f^1} \mathrm{Hom}_k(N, L) \otimes_A \mathrm{Hom}_k(M, N) \xrightarrow{\pi} \mathrm{Hom}_k(M, L),$$

where the last morphism is induced by composition. Since T is a t-algebra, we can identify $V \otimes_A V$ with $[T]_2$. Then the composition gf is defined, for any

$v \in V$, by the following

$$
\begin{aligned}
(gf)^0 &= g^0 f^0;\\
(gf)^1(v) &= g^0 f^1(v) + g^1(v) f^0 + \pi(g^1 \otimes f^1)(\delta(v)).
\end{aligned}
$$

*The full subcategory of $\mathcal{A}$-*Mod *consisting of all finite-dimensional objects will be denoted by $\mathcal{A}$-*mod.

Proposition 2.3. *Given a ditalgebra $\mathcal{A}$, the definition above indeed gives rise to a k-category $\mathcal{A}$-*Mod.

Proof. First we see that $gf = ((gf)^0, (gf)^1)$ is indeed a morphism. Clearly, $(gf)^0 \in \mathrm{Hom}_k(M, L)$. Let us verify that $(gf)^1 \in \mathrm{Hom}_{A\text{-}A}(V, \mathrm{Hom}_k(M, L))$. For this, take $v \in V$, $a \in A$ and $m \in M$, then

$$
\begin{aligned}
[(gf)^1(av)](m) &= [g^0 f^1(av)](m) + [g^1(av) f^0](m)\\
&\quad + [\pi(g^1 \otimes f^1)(\delta(av))](m)\\
&= ag^0[f^1(v)(m)] - g^1(\delta(a))[f^1(v)(m)] + [ag^1(v) f^0](m)\\
&\quad + [\pi(g^1 \otimes f^1)(\delta(a)v + a\delta(v))](m)\\
&= ag^0[f^1(v)(m)] - g^1(\delta(a))[f^1(v)(m)] + [ag^1(v) f^0](m)\\
&\quad + [g^1(\delta(a)) f^1(v)](m) + a[\pi(g^1 \otimes f^1)(\delta(v))](m)\\
&= ag^0[f^1(v)(m)] + [ag^1(v) f^0](m)\\
&\quad + a[\pi(g^1 \otimes f^1)(\delta(v))](m)\\
&= a[(gf)^1(v)](m).
\end{aligned}
$$

Now, take $a \in A$, $v \in V$ and $m \in M$, then

$$
\begin{aligned}
[(gf)^1(va)](m) &= [g^0 f^1(va)](m) + [g^1(va) f^0](m)\\
&\quad + [\pi(g^1 \otimes f^1)(\delta(va))](m)\\
&= [g^0(f^1(v)a)](m) + (g^1(v)a)[f^0(m)]\\
&\quad + [\pi(g^1 \otimes f^1)(\delta(v)a - v\delta(a))](m)\\
&= g^0[f^1(v)(am)] + (g^1(v))[af^0(m)]\\
&\quad + [\pi(g^1 \otimes f^1)(\delta(v)a - v\delta(a))](m)\\
&= g^0[f^1(v)(am)] + (g^1(v))[f^0(am) + f^1(\delta(a))(m)]\\
&\quad + [\pi(g^1 \otimes f^1)(\delta(v)a)](m) - [g^1(v) f^1(\delta(a))](m)\\
&= g^0[f^1(v)(am)] + (g^1(v))[f^0(am)]\\
&\quad + [\pi(g^1 \otimes f^1)(\delta(v)a)](m)\\
&= [g^0 f^1(v)](am) + [g^1(v) f^0](am)\\
&\quad + [\pi(g^1 \otimes f^1)(\delta(v))](am)\\
&= [(gf)^1(v)](am)\\
&= [(gf)^1(v)a](m).
\end{aligned}
$$

Finally, $gf \in \mathrm{Hom}_{\mathcal{A}}(M, L)$, because, for $a \in A$ and $m \in M$, we have

$$\begin{aligned}
[(gf)^0](am) &= [g^0 f^0](am) \\
&= g^0[af^0(m) - f^1(\delta(a))(m)] \\
&= g^0[af^0(m)] - g^0[f^1(\delta(a))](m) \\
&= ag^0[f^0(m)] - g^1(\delta(a))[f^0(m)] - g^0[f^1(\delta(a))](m) \\
&= [a(g^0 f^0)](m) \\
&\quad - [g^0 f^1(\delta(a)) + g^1(\delta(a)) f^0 + \pi(g^1 \otimes f^1)(\delta^2(a))][m] \\
&= [a(g^0 f^0)](m) - (gf)^1(\delta(a))[m] \\
&= [a(gf)^0](m) - (gf)^1(\delta(a))[m].
\end{aligned}$$

Clearly, for each $M \in \mathcal{A}$-Mod, the morphism $I_M = (I_M, 0)$ plays the role of an identity. Now we show that the composition is associative (it is clearly bilinear). Consider the morphisms $M \xrightarrow{f} N$, $N \xrightarrow{g} L$ and $L \xrightarrow{h} K$ in $\mathcal{A}$-Mod.

It is clear that $[h(gf)]^0 = [(hg)f]^0$. In order to show that $[h(gf)]^1 = [(hg)f]^1$, having in mind our identification $V \otimes_A V = [T]_2$, consider $v \in V$ and let $\delta(v) = \sum_a u_a \otimes w_a$, with $u_a, w_a \in V$. Assume that, for each a

$$\delta(u_a) = \sum_b u^1_{ab} \otimes u^2_{ab} \quad \text{and} \quad \delta(w_a) = \sum_c w^1_{ac} \otimes w^2_{ac}.$$

Then

$$\begin{aligned}
[h(gf)]^1(v) &= h^0(gf)^1(v) + h^1(v)(gf)^0 + \textstyle\sum_a h^1(u_a)(gf)^1(w_a) \\
&= h^0[g^0 f^1(v) + g^1(v) f^0 + \textstyle\sum_a g^1(u_a) f^1(w_a)] + h^1(v) g^0 f^0 \\
&\quad + \textstyle\sum_a h^1(u_a)[g^0 f^1(w_a) + g^1(w_a) f^0 + \sum_c g^1(w^1_{ac}) f^1(w^2_{ac})],
\end{aligned}$$

and

$$\begin{aligned}
[(hg)f]^1(v) &= (hg)^0 f^1(v) + (hg)^1(v) f^0 + \textstyle\sum_a (hg)^1(u_a) f^1(w_a) \\
&= h^0 g^0 f^1(v) + [h^0 g^1(v) + h^1(v) g^0 + \textstyle\sum_a h^1(u_a) g^1(w_a)] f^0 \\
&\quad + \textstyle\sum_a [h^0 g^1(u_a) + h^1(u_a) g^0 + \sum_b h^1(u^1_{ab}) g^1(u^2_{ab})] f^1(w_a).
\end{aligned}$$

Then, we have to show that

$$\sum_{a,c} h^1(u_a) g^1(w^1_{ac}) f^1(w^2_{ac}) = \sum_{a,b} h^1(u^1_{ab}) g^1(u^2_{ab}) f^1(w_a).$$

We have

$$\sum_a \delta(u_a) w_a + (-1)^{\deg(u_a)} u_a \delta(w_a) = \delta\left(\sum_a u_a \otimes w_a\right) = \delta^2(v) = 0,$$

which implies the following equality in $V \otimes_A V \otimes_A V = [T]_3$ (we use again that T is a t-algebra)

$$\sum_{a,c} u_a \otimes w^1_{ac} \otimes w^2_{ac} = \sum_{a,b} u^1_{ab} \otimes u^2_{ab} \otimes w_a.$$

Then, the following composition applied to the last equality gives the desired result

$$V\otimes_A V\otimes_A V\xrightarrow{h^1\otimes g^1\otimes f^1}\mathrm{Hom}_k(L,K)\otimes_A\mathrm{Hom}_k(N,L)\otimes_A\mathrm{Hom}_k(M,N)$$
$$\xrightarrow{\pi}\mathrm{Hom}_k(M,K).$$

□

Lemma 2.4. *Any morphism of ditalgebras $\phi:\mathcal{A}\longrightarrow\mathcal{A}'$ induces, by restriction, a functor $F_\phi:\mathcal{A}'$-Mod$\longrightarrow\mathcal{A}$-Mod. To give the explicit formula on morphisms, denote by $A=A_\mathcal{A}$ and $V=V_\mathcal{A}$, and with A' and V' the corresponding objects for the ditalgebra $\mathcal{A}'$; consider also the morphisms $\phi_0:A\to A'$ and $\phi_1:V\to V'$, induced by ϕ. Then, if $M\in\mathcal{A}'$-Mod, $F_\phi(M)$ is the A-module obtained from M by restriction of scalars through ϕ_0. The receipt on morphisms is given, for any $(f^0,f^1)\in\mathrm{Hom}_{\mathcal{A}'}(M,N)$, by $F_\phi(f^0,f^1)=(f^0,f^1\phi_1)$.*

If ϕ is surjective, then F_ϕ is faithful and injective on objects. Moreover, if $\phi':\mathcal{A}'\longrightarrow\mathcal{A}''$ is another morphism of ditalgebras, then $F_{\phi'\phi}=F_\phi F_{\phi'}$.

Proof. We first show that $F_\phi(f^0,f^1)\in\mathrm{Hom}_\mathcal{A}(F_\phi(M),F_\phi(N))$, whenever we have $(f^0,f^1)\in\mathrm{Hom}_{\mathcal{A}'}(M,N)$. For $m\in M$ and $a\in A$, we have

$$\begin{aligned}F_\phi(f)^0(am)&=f^0(am)\\&=f^0[\phi_0(a)m]\\&=\phi_0(a)f^0(m)-f^1(\delta'(\phi_0(a)))[m]\\&=af^0(m)-f^1\phi_1(\delta(a))[m]\\&=aF_\phi(f)^0[m]-F_\phi(f)^1(\delta(a))[m].\end{aligned}$$

In order to show that F_ϕ preserves the composition, take $f\in\mathrm{Hom}_{\mathcal{A}'}(M,N)$ and $g\in\mathrm{Hom}_{\mathcal{A}'}(N,L)$. Therefore, $[F_\phi(gf)]^0=(gf)^0=g^0f^0=[F_\phi(g)F_\phi(f)]^0$. Moreover, for $v\in V$ with $\delta(v)=\sum_i v_i^1\otimes v_i^2$, we have $\delta'(\phi(v))=\phi\delta(v)=\sum_i\phi(v_i^1)\otimes\phi(v_i^2)$. Therefore

$$\begin{aligned}\left[F_\phi(g)F_\phi(f)\right]^1(v)&=F_\phi(g)^0F_\phi(f)^1(v)+F_\phi(g)^1(v)F_\phi(f)^0\\&\quad+\pi(F_\phi(g)^1\otimes F_\phi(f)^1)(\delta(v))\\&=g^0f^1(\phi(v))+g^1(\phi(v))f^0+\textstyle\sum_i g^1(\phi(v_i^1))f^1(\phi(v_i^2))\\&=g^0f^1(\phi(v))+g^1(\phi(v))f^0+\pi(g^1\otimes f^1)(\delta'(\phi(v)))\\&=(gf)^1(\phi(v))\\&=\left[F_\phi(gf)\right]^1(v).\end{aligned}$$

We have seen that $F_\phi(gf)=F_\phi(g)F_\phi(f)$. Clearly, F_ϕ preserves identities. □

Remark 2.5. *Whenever $\mathcal{A}$ is a ditalgebra, there is a canonical embedding*

$$L=L_\mathcal{A}:A\text{-Mod}\longrightarrow\mathcal{A}\text{-Mod},$$

which is the identity on objects, and satisfies $L(f^0) = (f^0, 0)$*, whenever* $f^0 \in \mathrm{Hom}_A(M, N)$*. As a consequence,* $\mathcal{A}$-Mod *is an additive category: Given* M_1, M_2 *in* $\mathcal{A}$-Mod*, their direct sum in* $\mathcal{A}$-Mod *is* $\{M_i \xrightarrow{(\sigma_i,0)} M_1 \oplus M_2 \xrightarrow{(\pi_i,0)} M_i\}_{i=1,2}$*, where* $\{M_i \xrightarrow{\sigma_i} M_1 \oplus M_2 \xrightarrow{\pi_i} M_i\}_{i=1,2}$ *is the direct sum of* M_1 *and* M_2 *in* A-Mod*.*

As in any additive category, each morphism $f \in \mathrm{Hom}_{\mathcal{A}}(M_1 \oplus M_2, N_1 \oplus N_2)$ *can be written as a matrix* $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ *of morphisms* $f_{ji} = (\pi_j, 0) f (\sigma_i, 0) : M_i \longrightarrow N_j$*. In our case, the explicit description of* f_{ji} *is given by* $f^0_{ji} = \pi_j f^0 \sigma_i$*, and, for* $v \in V$*,* $f^1_{ji}(v) = \pi_j f^1(v)\sigma_i$*. And, as usual, if* $f \in \mathrm{Hom}_{\mathcal{A}}(M_1, N_1)$ *and* $g \in \mathrm{Hom}_{\mathcal{A}}(M_2, N_2)$*, with* $f \oplus g$*, we denote the direct sum morphism* $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \mathrm{Hom}_{\mathcal{A}}(M_1 \oplus M_2, N_1 \oplus N_2)$*.*

Remark 2.6. *If* Λ *is a* k*-algebra, the associated regular ditalgebra is* $\mathcal{R}_\Lambda = (\Lambda, 0)$*, where* $[\Lambda]_0 = \Lambda$*. Clearly* $\mathcal{R}_\Lambda$-Mod $\cong \Lambda$-Mod*. Given a ditalgebra* $\mathcal{A} = (T, \delta)$*, the projection* $T \longrightarrow [T]_0 = A$ *determines a morphism of ditalgebras* $\pi : \mathcal{A} \longrightarrow \mathcal{R}_A$*. The induced functor* $F_\pi : A$-Mod$\longrightarrow \mathcal{A}$-Mod *is the canonical embedding* $L_{\mathcal{A}}$ *of (2.5).*

Definition 2.7. *Let* $\mathcal{A} = (T, \delta)$ *be any ditalgebra. Then, its opposite ditalgebra* $\mathcal{A}^{op} = (T^{op}, \delta^{op})$ *is constituted by the opposite algebra* T^{op} *of* T*, where* $v \cdot^{op} u = uv$*, with the same grading and* $\delta^{op}(t) = (-1)^{\deg(t)}\delta(t)$*, for any homogeneous* $t \in T$*. In particular,* $\delta(a) = \delta^{op}(a)$*, for any* $a \in A$*, and* $\delta^{op}(v) = -\sum_i v_i \cdot^{op} u_i$*, whenever* $\delta(v) = \sum_i u_i v_i$ *with* $v, v_i, u_i \in V$*.*

Proposition 2.8. *Let* $\mathcal{A}$ *be any ditalgebra. Then, the functor* $D = \mathrm{Hom}_A(-, k)$: k-Mod$\longrightarrow k$-Mod *permits to define a contravariant* k*-functor*

$$D : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}^{op}\text{-Mod}$$

where, given $f = (f^0, f^1) \in \mathrm{Hom}_{\mathcal{A}}(M, N)$*,* $D(f) := (D(f^0), -D(f^1))$*, with* $D(f^1)[v] = D(f^1[v]) \in \mathrm{Hom}_k(D(N), D(M))$*, for any* $v \in V$*. Moreover, if we restrict ourselves to finite-dimensional modules, we obtain a duality*

$$D : \mathcal{A}\text{-mod} \longrightarrow \mathcal{A}^{op}\text{-mod} \quad \textit{with} \quad D^2 \cong Id.$$

Proof. We first show that $D(f) \in \mathrm{Hom}_{\mathcal{A}^{op}}(D(N), D(M))$. It is clear that $D(f^0) \in \mathrm{Hom}_k(D(N), D(M))$. We have that $A_{\mathcal{A}^{op}} = [T^{op}]_0 = A^{op}$ and $V^{op} := V_{\mathcal{A}^{op}} = [T^{op}]_1$ has the same underlying vectorspace V, but its A^{op}-bimodule action is related to the A-bimodule action on V by the formula $a^{op} v b^{op} = bva$, where we agree to write a^{op} for the element a of A when considered as an element of A^{op}. If we consider left A^{op}-modules as right A-modules, then

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Excerpt
More Information

$\mathrm{Hom}_{A^{op}\text{-}A^{op}}(V^{op}, \mathrm{Hom}_k(D(N), D(M))) = \mathrm{Hom}_{A\text{-}A}(V, \mathrm{Hom}_k(D(N), D(M)))$. We want to show that $D(f^1) \in \mathrm{Hom}_{A\text{-}A}(V, \mathrm{Hom}_k(D(N), D(M)))$. Let $v \in V$, $a \in A$, $\phi \in D(N)$ and $m \in M$, then

$$\begin{aligned}\big(D(f^1)[va]\big)(\phi)[m] &= \phi(f^1(va)[m])\\ &= \phi((f^1(v)a)[m])\\ &= \phi(f^1(v)[am])\\ &= (\phi f^1(v))[am]\\ &= (\phi f^1(v))a[m]\\ &= \big[\big(D(f^1)[v]\big)(\phi)\big]a[m]\\ &= \big[\big(D(f^1)[v]\big)a\big](\phi)[m].\end{aligned}$$

Moreover

$$\begin{aligned}\big(D(f^1)[av]\big)(\phi)[m] &= \phi(f^1(av)[m])\\ &= \phi((af^1(v))[m])\\ &= \phi(af^1(v)[m])\\ &= (\phi a)[f^1(v)[m]]\\ &= (\phi a)f^1(v)[m]\\ &= \big[\big(D(f^1)[v]\big)(\phi a)\big][m]\\ &= a\big[D(f^1)[v]\big](\phi)[m].\end{aligned}$$

Now we verify that $D(f)$ is a morphism, which means that, for $a \in A$ and $\phi \in D(N)$, $([D(f^0)](\phi))a = (D(f^0))(\phi a) - D(f^1)[\delta(a)](\phi)$. Take $m \in M$, then

$$\begin{aligned}\big[\big([D(f^0)](\phi)\big)a\big][m] &= \big(\phi f^0\big)a[m]\\ &= (\phi f^0)[am]\\ &= \phi(f^0[am])\\ &= \phi\big(af^0[m] - f^1(\delta(a))[m]\big)\\ &= (\phi a)(f^0[m]) - \big(\phi f^1(\delta(a))\big)[m]\\ &= \big(D(f^0)\big)(\phi a)[m] - D(f^1)[\delta(a)](\phi)[m].\end{aligned}$$

It is clear that D preserves identities; now we check that D reverses composition. Take $g \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ and $f \in \mathrm{Hom}_{\mathcal{A}}(N, L)$. We have to verify that each component of $D(fg)$ and $D(g)D(f)$ coincide. This is clear for the first one. Take $v \in V^{op}$, then

$$\begin{aligned}-D(fg)^1(v) &= -D\left[f^0g^1(v) + f^1(v)g^0 + \pi(f^1 \otimes g^1)\delta(v)\right]\\ &= -D(g^1(v))D(f^0) - D(g^0)D(f^1(v)) - D\left[\pi(f^1 \otimes g^1)\delta(v)\right].\end{aligned}$$

Moreover

$$\begin{aligned}[D(g)D(f)]^1(v) = {}& -D(g^0)D(f^1(v)) - D(g^1(v))D(f^0)\\ & + \widehat{\pi}(D(g^1) \otimes D(f^1))\delta^{op}(v).\end{aligned}$$

We are referring to the following maps

$$V \xrightarrow{\delta} [T]_2 = V \otimes_A V, \qquad V^{op} \xrightarrow{\delta^{op}} [T^{op}]_2 = V^{op} \otimes_{A^{op}} V^{op},$$

$$V \otimes_A V \xrightarrow{f^1 \otimes g^1} \mathrm{Hom}_k(N, L) \otimes_A \mathrm{Hom}_k(M, N) \xrightarrow{\pi} \mathrm{Hom}_k(M, L), \text{ and}$$

$$V^{op} \otimes_{A^{op}} V^{op} \xrightarrow{D(g^1) \otimes D(f^1)} \mathrm{Hom}_k(DN, DM) \otimes_{A^{op}} \mathrm{Hom}_k(DL, DN)$$

$$\xrightarrow{\widehat{\pi}} \mathrm{Hom}_k(DL, DM),$$

where π and $\widehat{\pi}$ denote composition. Assume that $\delta(v) = \sum_i u_i \otimes v_i$, with $v, u_i, v_i \in V$, then $\delta^{op}(v) = -\sum_i v_i \otimes u_i$. Hence

$$\begin{aligned} -D\left[\pi(f^1 \otimes g^1)\delta(v)\right] &= -D\left[\textstyle\sum_i f^1(u_i) g^1(v_i)\right] \\ &= -\textstyle\sum_i D(g^1)(v_i) D(f^1)(u_i) \\ &= \widehat{\pi}(D(g^1) \otimes D(f^1))\delta^{op}(v). \end{aligned}$$

Then, we have a contravariant k-functor $D : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}^{op}\text{-Mod}$. Now consider its restriction to finite-dimensional modules $D : \mathcal{A}\text{-mod} \longrightarrow \mathcal{A}^{op}\text{-mod}$. Let us show an isomorphism of functors $\eta : Id \longrightarrow D^2$. We can consider the standard duality $D : A\text{-mod} \longrightarrow A^{op}\text{-mod}$ and we know that the evaluation map $\eta^0_M : M \longrightarrow D^2 M$ yields an isomorphism of functors $\eta^0 : Id_{A\text{-mod}} \longrightarrow D^2$. Then, we have an isomorphism of functors $\eta : Id_{\mathcal{A}\text{-mod}} \longrightarrow D^2$, defined by $\eta_M = (\eta^0_M, 0)$, for each object $M \in \mathcal{A}\text{-mod}$. □

Exercise 2.9. *Let $\{M_i\}_{i \in I}$ be a family of modules of the ditalgebra $\mathcal{A}$ and write $A := A_{\mathcal{A}}$. Consider the product $P := \prod_i M_i$ of the family in A-Mod, with associated projections $\{\pi^0_i : P \longrightarrow M_i\}_{i \in I}$, and the coproduct $Q := \coprod_i M_i$ of the family in A-Mod, with associated injections $\{\sigma^0_i : M_i \longrightarrow Q\}_{i \in I}$. Show that:*

(1) P is the product of the family $\{M_i\}_{i \in I}$ in the category $\mathcal{A}$-Mod, with associated projections $\{\pi_i := (\pi^0_i, 0) : P \longrightarrow M_i\}_{i \in I}$.
(2) Q is the coproduct of the family $\{M_i\}_{i \in I}$ in the category $\mathcal{A}$-Mod, with associated injections $\{\sigma_i := (\sigma^0_i, 0) : M_i \longrightarrow Q\}_{i \in I}$.
(3) Therefore, for $M, N \in \mathcal{A}$-Mod, we have canonical isomorphisms

$$\begin{cases} \mathrm{Hom}_{\mathcal{A}}(\coprod_i M_i, N) \cong \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(M_i, N), & \text{given by } g \mapsto (g\sigma_i), \text{ and} \\ \mathrm{Hom}_{\mathcal{A}}(M, \prod_i M_i) \cong \prod_{i \in I} \mathrm{Hom}_{\mathcal{A}}(M, M_i), & \text{given by } f \mapsto (\pi_i f). \end{cases}$$

Exercise 2.10. *Let $\mathcal{A} = (T, \delta)$ be any k-ditalgebra and write $A = [T]_0$ and $V = [T]_1$. Given $M, N \in \mathcal{A}$-Mod, we have canonical morphisms of k-algebras $\psi_M : A \longrightarrow \mathrm{End}_k(M)$ and $\psi_N : A \longrightarrow \mathrm{End}_k(N)$, given by the corresponding A-module structures on the k-vector spaces M and N. For any pair $g = (g^0, g^1) \in \mathrm{Hom}_k(M, N) \times \mathrm{Hom}_{A\text{-}A}(V, \mathrm{Hom}_k(M, N))$, we can consider*