Notes on Sela's work: Limit groups and Makanin-Razborov diagrams

Mladen Bestvina* Mark Feighn*

Abstract

This is the first in a planned series of papers giving an alternate approach to Zlil Sela's work on the Tarski problems. The present paper is an exposition of work of Kharlampovich-Myasnikov and Sela giving a parametrization of $Hom(G, \mathbb{F})$ where G is a finitely generated group and $\mathbb{F}$ is a non-abelian free group.

Contents

1 The Main Theorem

1.1 Introduction

This is the first of a planned series of papers giving an alternative approach to Zlil Sela's work on the Tarski problems [35, 34, 36, 38, 37, 39, 40, 41, 31, 32].

*The authors gratefully acknowledge support of the National Science Foundation. This material is based upon work supported by the National Science Foundation under Grants Nos. DMS-0502441 and DMS-0805440.

The present paper is an exposition of the following result of Kharlampovich-Myasnikov [14, 15] and Sela [34]:

Theorem. *Let G be a finitely generated non-free group. There is a finite collection $\{q_i : G \to \Gamma_i\}$ of proper epimorphisms of G such that, for any homomorphism f from G to a free group F, there is $\alpha \in Aut(G)$ such that $f\alpha$ factors through some q_i.*

A more refined statement is given in the Main Theorem on page 7. Our approach, though similar to Sela's, differs in several aspects: notably a different measure of complexity and a more geometric proof which avoids the use of the full Rips theory for finitely generated groups acting on $\mathbb{R}$-trees; see Section 7. We attempted to include enough background material to make the paper self-contained. See Paulin [24] and Champetier-Guirardel [5] for accounts of some of Sela's work on the Tarski problems.

The first version of these notes was circulated in 2003. In the meantime Henry Wilton [45] made available solutions to the exercises in the notes. We also thank Wilton for making numerous comments that led to many improvements.

Remark 1.1. In the theorem above, since G is finitely generated we may assume that F is also finitely generated. If F is abelian, then any f factors through the abelianization of G mod its torsion subgroup and we are in the situation of Example 1.4 below. Finally, if F_1 and F_2 are finitely generated non-abelian free groups then there is an injection $F_1 \to F_2$. So, if $\{q_i\}$ is a set of epimorphisms that satisfies the conclusion of the theorem for maps to F_2, then $\{q_i\}$ also works for maps to F_1. Therefore, throughout the paper we work with a fixed finitely generated non-abelian free group $\mathbb{F}$.

Notation 1.2. Finitely generated (finitely presented) is abbreviated fg (respectively fp).

The main goal of [34] is to give an answer to the following:

Question 1. *Let G be an fg group. Describe the set of all homomorphisms from G to $\mathbb{F}$.*

Example 1.3. When G is a free group, we can identify $Hom(G, \mathbb{F})$ with the cartesian product $\mathbb{F}^n$ where $n = rank(G)$.

Example 1.4. If $G = \mathbb{Z}^n$, let $\mu : \mathbb{Z}^n \to \mathbb{Z}$ be the projection to one of the coordinates. If $h : \mathbb{Z}^n \to \mathbb{F}$ is a homomorphism, there is an automorphism $\alpha : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $h\alpha$ factors through μ. This provides an explicit (although not 1-1) parametrization of $Hom(G, \mathbb{F})$ by $Aut(\mathbb{Z}^n) \times Hom(\mathbb{Z}, \mathbb{F}) \cong GL_n(\mathbb{Z}) \times \mathbb{F}$.

Example 1.5. When G is the fundamental group of a closed genus g orientable surface, let $\mu : G \to F_g$ denote the homomorphism to a free group of rank g induced by the (obvious) retraction of the surface to the rank g graph. It is a folk theorem[1] that for every homomorphism $f : G \to \mathbb{F}$ there is an automorphism $\alpha : G \to G$ (induced by a homeomorphism of the surface) so that $f\alpha$ factors through μ. The theorem was generalized to the case when G is the fundamental group of a non-orientable closed surface by Grigorchuk and Kurchanov [9]. Interestingly, in this generality the single map μ is replaced by a finite collection $\{\mu_1, \cdots, \mu_k\}$ of maps from G to a free group F. In other words, for all $f \in Hom(G, \mathbb{F})$ there is $\alpha \in Aut(G)$ induced by a homeomorphism of the surface such that $f\alpha$ factors through some μ_i.

1.2 Basic properties of limit groups

Another goal is to understand the class of groups that naturally appear in the answer to the above question, these are called limit groups.

Definition 1.6. Let G be an fg group. A sequence $\{f_i\}$ in $Hom(G, \mathbb{F})$ is *stable* if, for all $g \in G$, the sequence $\{f_i(g)\}$ is eventually always 1 or eventually never 1. The *stable kernel* of $\{f_i\}$, denoted $\underrightarrow{Ker}\ f_i$, is

$$\{g \in G \mid f_i(g) = 1 \text{ for almost all } i\}.$$

An fg group Γ is a *limit group* if there is an fg group G and a stable sequence $\{f_i\}$ in $Hom(G, \mathbb{F})$ so that $\Gamma \cong G/\underrightarrow{Ker}\ f_i$.

Remark 1.7. One can view each f_i as inducing an action of G on the Cayley graph of $\mathbb{F}$, and then can pass to a limiting $\mathbb{R}$-tree action (after a subsequence). If the limiting tree is not a line, then $\underrightarrow{Ker}\ f_i$ is precisely the kernel of this action and so Γ acts faithfully. This explains the name.

Definition 1.8. An fg group Γ is *residually free* if for every element $\gamma \in \Gamma$ there is $f \in Hom(\Gamma, \mathbb{F})$ such that $f(\gamma) \neq 1$. It is *ω-residually free* if for every finite subset $X \subset \Gamma$ there is $f \in Hom(\Gamma, \mathbb{F})$ such that $f|X$ is injective.

Exercise 2. *Residually free groups are torsion free.*

Exercise 3. *Free groups and free abelian groups are ω-residually free.*

Exercise 4. *The fundamental group of $n\mathbb{P}^2$ for $n = 1$, 2, or 3 is not ω-residually free, see [18].*

Exercise 5. *Every ω-residually free group is a limit group.*

Exercise 6. *An fg subgroup of an ω-residually free group is ω-residually free.*

Exercise 7. *Every non-trivial abelian subgroup of an ω-residually free group is contained in a unique maximal abelian subgroup. For example, $F \times \mathbb{Z}$ is not ω-residually free for any non-abelian F.*

[1] see Zieschang [46] and Stallings [43]

Lemma 1.9. *Let $G_1 \to G_2 \to \cdots$ be an infinite sequence of epimorphisms between fg groups. Then the sequence*

$$Hom(G_1, \mathbb{F}) \leftarrow Hom(G_2, \mathbb{F}) \leftarrow \cdots$$

eventually stabilizes (consists of bijections).

Proof. Embed $\mathbb{F}$ as a subgroup of $SL_2(\mathbb{R})$. That the corresponding sequence of varieties $Hom(G_i, SL_2(\mathbb{R}))$ stabilizes follows from algebraic geometry, and this proves the lemma. □

Corollary 1.10. *A sequence of epimorphisms between $(\omega-)$residually free groups eventually stabilizes.* □

Lemma 1.11. *Every limit group is ω-residually free.*

Proof. Let Γ be a limit group, and let G and $\{f_i\}$ be as in the definition. Without loss, G is fp. Now consider the sequence of quotients

$$G \to G_1 \to G_2 \to \cdots \to \Gamma$$

obtained by adjoining one relation at a time. If Γ is fp the sequence terminates, and in general it is infinite. Let $G' = G_j$ be such that $Hom(G', \mathbb{F}) = Hom(\Gamma, \mathbb{F})$. All but finitely many f_i factor through G' since each added relation is sent to 1 by almost all f_i. It follows that these f_i factor through Γ and each non-trivial element of Γ is sent to 1 by only finitely many f_i. By definition, Γ is ω-residually free. □

The next two exercises will not be used in this paper but are included for their independent interest.

Exercise 8. *Every ω-residually free group Γ embeds into $PSL_2(\mathbb{R})$, and also into $SO(3)$.*

Exercise 9. *Let Γ be ω-residually free. For any finite collection of nontrivial elements $g_1, \cdots, g_k \in \Gamma$ there is an embedding $\Gamma \to PSL_2(\mathbb{R})$ whose image has no parabolic elements and so that $g_1, \cdots, g_k$ go to hyperbolic elements.*

1.3 Modular groups and the statement of the main theorem

Only certain automorphisms, called *modular automorphisms*, are needed in the theorem on page 2. This section contains a definition of these automorphisms.

Definition 1.12. Free products with amalgamations and HNN-decompositions of a group G give rise to *Dehn twist automorphisms of* G. Specifically, if $G = A *_C B$ and if z is in the centralizer $Z_B(C)$ of C in B, then the automorphism α_z of G, called the *Dehn twist in* z, is determined as follows.

$$\alpha_z(g) = \begin{cases} g, & \text{if } g \in A; \\ zgz^{-1}, & \text{if } g \in B. \end{cases}$$

If $C \subset A$, $\phi : C \to A$ is a monomorphism, $G = A*_C = \langle A, t \mid tat^{-1} = \phi(a), a \in A\rangle$,[2] and $z \in Z_A(C)$, then α_z is determined as follows.

$$\alpha_z(g) = \begin{cases} g, & \text{if } g \in A; \\ gz, & \text{if } g = t. \end{cases}$$

Definition 1.13. A GAD[3] of a group G is a finite graph of groups decomposition[4] of G with abelian edge groups in which some of the vertices are designated QH[5] and some others are designated *abelian*, and the following holds.

- A QH-vertex group is the fundamental group of a compact surface S with boundary and the boundary components correspond to the incident edge groups (they are all infinite cyclic). Further, S carries a pseudoAnosov homeomorphism (so S is a torus with 1 boundary component or $\chi(S) \leq -2$).

- An abelian vertex group A is non-cyclic abelian. Denote by $P(A)$ the subgroup of A generated by incident edge groups. The *peripheral subgroup of* A, denoted $\overline{P}(A)$, is the subgroup of A that dies under every homomorphism from A to $\mathbb{Z}$ that kills $P(A)$, i.e.

$$\overline{P}(A) = \cap\{Ker(f) \mid f \in Hom(A, \mathbb{Z}), P(A) \subset Ker(f)\}.$$

The non-abelian non-QH vertices are *rigid*.

Remark 1.14. We allow the possibility that edge and vertex groups of GAD's are not fg.

Remark 1.15. If Δ is a GAD for a fg group G, and if A is an abelian vertex group of Δ, then there are epimorphisms $G \to A/P(A) \to A/\overline{P}(A)$. Hence, $A/P(A)$ and $A/\overline{P}(A)$ are fg. Since $A/\overline{P}(A)$ is also torsion free, $A/\overline{P}(A)$ is free,

[2] t is called a *stable letter*.
[3] Generalized Abelian Decomposition
[4] We will use the terms *graph of groups decomposition* and *splitting* interchangeably. Without further notice, splittings are always *minimal*, i.e. the associated G-tree has no proper invariant subtrees.
[5] Quadratically Hanging

and so $A = A_0 \oplus \overline{P}(A)$ with $A_0 \cong A/\overline{P}(A)$ a retract of G. Similarly, $A/\overline{P}(A)$ is a direct summand of $A/P(A)$. A summand complementary to $A/\overline{P}(A)$ in $A/P(A)$ must be a torsion group by the definition of $\overline{P}(A)$. In particular, $P(A)$ has finite index in $\overline{P}(A)$. It also follows from the definition of $\overline{P}(A)$ that any automorphism leaving $P(A)$ invariant must leave $\overline{P}(A)$ invariant as well. It follows that if A is torsion free, then any automorphism of A that is the identity when restricted to $P(A)$ is also the identity when restricted to $\overline{P}(A)$.

Definition 1.16. The *modular group* $Mod(\Delta)$ associated to a GAD Δ of G is the subgroup of $Aut(G)$ generated by

- inner automorphisms of G,
- Dehn twists in elements of G that centralize an edge group of Δ,
- unimodular[6] automorphisms of an abelian vertex group that are the identity on its peripheral subgroup and all other vertex groups, and
- automorphisms induced by homeomorphisms of surfaces S underlying QH-vertices that fix all boundary components. If S is closed and orientable, we require the homeomorphisms to be orientation-preserving[7].

The *modular group of* G, denoted $Mod(G)$, is the subgroup of $Aut(G)$ generated by $Mod(\Delta)$ for all GAD's Δ of G. At times it will be convenient to view $Mod(G)$ as a subgroup of $Out(G)$. In particular, we will say that an element of $Mod(G)$ is *trivial* if it is an inner automorphism.

Definition 1.17. A *generalized Dehn twist* is a Dehn twist or an automorphism α of $G = A *_C B$ or $G = A*_C$ where in each case A is abelian, α restricted to $\overline{P}(A)$ and B is the identity, and α induces a unimodular automorphism of $A/\overline{P}(A)$. Here $\overline{P}(A)$ is the peripheral subgroup of A when we view $A *_C B$ or $G = A$ as a GAD with one or zero edges and abelian vertex A. If C is an edge groups of a GAD for G and if $z \in Z_G(C)$, then C determines a splitting of G as above and so also a Dehn twist in z. Similarly, an abelian vertex A of a GAD determines[8] a splitting $A *_C B$ and so also generalized Dehn twists.

Exercise 10. *$Mod(G)$ is generated by inner automorphisms together with generalized Dehn twists.*

Definition 1.18. A *factor set* for a group G is a finite collection of proper epimorphisms $\{q_i : G \to G_i\}$ such that if $f \in Hom(G, \mathbb{F})$ then there is $\alpha \in Mod(G)$ such that $f\alpha$ factors through some q_i.

[6] The induced automorphism of $A/\overline{P}(A)$ has determinant 1.
[7] We will want our homeomorphisms to be products of Dehn twists.
[8] by folding together the edges incident to A

Main Theorem ([14, 15, 35]). *Let G be an fg group that is not free. Then, G has a factor set $\{q_i : G \to \Gamma_i\}$ with each Γ_i a limit group. If G is not a limit group, we can always take α to be the identity.*

We will give two proofs–one in Section 4 and the second, which uses less in the way of technical machinery, in Section 7. In the remainder of this section, we explore some consequences of the Main Theorem and then give another description of limit groups.

1.4 Makanin-Razborov diagrams

Corollary 1.19. *Iterating the construction of the Main Theorem (for Γ_i's etc.) yields a finite tree of groups terminating in groups that are free.*

Proof. If $\Gamma \to \Gamma'$ is a proper epimorphism between limit groups, then since limit groups are residually free, $Hom(\Gamma', \mathbb{F}) \subsetneq Hom(\Gamma, \mathbb{F})$. We are done by Lemma 1.9. □

Definition 1.20. The tree of groups and epimorphisms provided by Corollary 1.19 is called an *MR-diagram*[9] for G (with respect to $\mathbb{F}$). If

$$G \xrightarrow{q} \Gamma_1 \xrightarrow{q_1} \Gamma_2 \xrightarrow{q_2} \cdots \xrightarrow{q_{m-1}} \Gamma_m$$

is a branch of an MR-diagram and if $f \in Hom(G, \mathbb{F})$ then we say that f *MR-factors* through this branch if there are $\alpha \in Mod(G)$ (which is the identity if G is not a limit group), $\alpha_i \in Mod(\Gamma_i)$, for $1 \le i < m$, and $f' \in Hom(\Gamma_m, \mathbb{F})$ (recall Γ_m is free) such that $f = f' q_{m-1} \alpha_{m-1} \cdots q_1 \alpha_1 q \alpha$.

Remark 1.21. The key property of an MR-diagram for G is that, for $f \in Hom(G, \mathbb{F})$, there is a branch of the diagram through which f MR-factors. This provides an answer to Question 1 in that $Hom(G, \mathbb{F})$ is parametrized by branches of an MR-diagram and, for each branch as above, $Mod(G) \times Mod(\Gamma_1) \times \cdots \times Mod(\Gamma_{m-1}) \times Hom(\Gamma_m, \mathbb{F})$. Note that if Γ_m has rank n, then $Hom(\Gamma_m, \mathbb{F}) \cong \mathbb{F}^n$.

In [32], Sela constructed MR-diagrams with respect to hyperbolic groups. In her thesis [1], Emina Alibegović constructed MR-diagrams with respect to limit groups. More recently, Daniel Groves [10, 11] constructed MR-diagrams with respect to torsion-free groups that are hyperbolic relative to a collection of free abelian subgroups.

1.5 Abelian subgroups of limit groups

Corollary 1.22. *Abelian subgroups of limit groups are fg and free.*

[9] for Makanin-Razborov, cf. [19, 20, 25].

Along with the Main Theorem, the proof of Corollary 1.22 will depend on an exercise and two lemmas.

Exercise 11 ([34, Lemma 2.3])**.** *Let M be a non-cyclic maximal abelian subgroup of the limit group Γ.*

1. *If $\Gamma = A *_C B$ with C abelian, then M is conjugate into A or B.*
2. *If $\Gamma = A*_C$ with C abelian, then either M is conjugate into A or there is a stable letter t such that M is conjugate to $M' = \langle C, t\rangle$ and $\Gamma = A *_C M'$.*

As a consequence, if $\alpha \in Mod(\Gamma)$ is a generalized Dehn twist and $\alpha|M$ is non-trivial, then there is an element $\gamma \in \Gamma$ and a GAD $\Delta = M *_C B$ or $\Delta = M$ for Γ such that, up to conjugation by γ, α is induced by a unimodular automorphism of $M/\overline{P}(M)$ (as in Definition 1.17). (Hint: Use Exercise 7.)

Lemma 1.23. *Suppose that Γ is a limit group with factor set $\{q_i : \Gamma \to G_i\}$. If H is a (not necessarily fg) subgroup of Γ such that, for every homomorphism $f : \Gamma \to \mathbb{F}$, $f|H$ factors through some $q_i|H$ (pre-compositions by automorphisms of Γ not needed) then, for some i, $q_i|H$ is injective.*

Proof. Suppose not and let $1 \neq h_i \in Ker(q_i|H)$. Since Γ is a limit group, there is $f \in Hom(\Gamma, \mathbb{F})$ that is injective on $\{1, h_1, \cdots, h_n\}$. On the other hand, $f|H$ factors through some $q_i|H$ and so $h_i = 1$, a contradiction. □

Lemma 1.24. *Let M be a non-cyclic maximal abelian subgroup of the limit group Γ. There is an epimorphism $r : \Gamma \to A$ where A is free abelian and every modular automorphism of Γ is trivial*[10] *when restricted to $M \cap Ker(r)$.*

Proof. By Exercise 10, it is enough to find r such that $\alpha|M \cap Ker(r)$ is trivial for every generalized Dehn twist $\alpha \in Mod(\Gamma)$. By Exercise 11 and Remark 1.15, there is a fg free abelian subgroup M_α of M and a retraction $r_\alpha : \Gamma \to M_\alpha$ such that $\alpha|M \cap Ker(r_\alpha)$ is trivial. Let $r = \Pi_\alpha r_\alpha : \Gamma \to \Pi_\alpha M_\alpha$ and let A be the image of r. Since Γ is fg, so is A. Hence A is free abelian. □

Proof of Corollary 1.22. Let M be a maximal abelian subgroup of a limit group Γ. We may assume that M is not cyclic. Since Γ is torsion free, it is enough to show that M is fg. By restricting the map r of Lemma 1.24 to M, we see that $M = A \oplus A'$ where A is fg and each $\alpha|A'$ is trivial. Let $\{q_i : \Gamma \to \Gamma_i\}$ be a factor set for Γ given by Theorem 1.3. By Lemma 1.23, A' injects into some Γ_i. Since $Hom(\Gamma_i, \mathbb{F}) \subsetneq Hom(\Gamma, \mathbb{F})$, we may conclude by induction that A' and hence M is fg. □

[10] agrees with the restriction of an inner automorphism of Γ.

1.6 Constructible limit groups

It will turn out that limit groups can be built up inductively from simpler limit groups. In this section, we give this description and list some properties that follow.

Definition 1.25. We define a hierarchy of fg groups – if a group belongs to this hierarchy it is called a CLG[11].

Level 0 of the hierarchy consists of fg free groups.

A group Γ belongs to level $\leq n+1$ iff either it has a free product decomposition $\Gamma = \Gamma_1 * \Gamma_2$ with Γ_1 and Γ_2 of level $\leq n$ or it has a homomorphism $\rho : \Gamma \to \Gamma'$ with Γ' of level $\leq n$ and it has a GAD such that

- ρ is injective on the peripheral subgroup of each abelian vertex group.
- ρ is injective on each edge group E and at least one of the images of E in a vertex group of the one-edged splitting induced by E is a maximal abelian subgroup.
- The image of each QH-vertex group is a non-abelian subgroup of Γ'.
- For every rigid vertex group B, ρ is injective on the *envelope* $\tilde{B}$ *of* B, defined by first replacing each abelian vertex with the peripheral subgroup and then letting $\tilde{B}$ be the subgroup of the resulting group generated by B and by the centralizers of incident edge-groups.

Example 1.26. A fg free abelian group is a CLG of level one (consider a one-point GAD for $\mathbb{Z}^n$ and $\rho : \mathbb{Z}^n \to \langle 0 \rangle$). The fundamental group of a closed surface S with $\chi(S) \leq -2$ is a CLG of level one. For example, an orientable genus 2 surface is a union of 2 punctured tori and the retraction to one of them determines ρ. Similarly, a non-orientable genus 2 surface is the union of 2 punctured Klein bottles.

Example 1.27. Start with the circle and attach to it 3 surfaces with one boundary component, with genera 1, 2, and 3 say. There is a retraction to the surface of genus 3 that is the union of the attached surfaces of genus 1 and 2. This retraction sends the genus 3 attached surface say to the genus 2 attached surface by “pinching a handle”. The GAD has a central vertex labeled $\mathbb{Z}$ and there are 3 edges that emanate from it, also labeled $\mathbb{Z}$. Their other endpoints are QH-vertex groups. The map induced by retraction satisfies the requirements so the fundamental group of the 2-complex built is a CLG.

Example 1.28. Choose a primitive[12] w in the fg free group F and form $\Gamma = F *_{\mathbb{Z}} F$, the *double of* F *along* $\langle w \rangle$ (so $1 \in \mathbb{Z}$ is identified with w on both sides). There is a retraction $\Gamma \to F$ that satisfies the requirements (both vertices are rigid), so Γ is a CLG.

[11] Constructible Limit Group
[12] no proper root

The following can be proved by induction on levels.

Exercise 12. *Every CLG is fp, in fact coherent. Every fg subgroup of a CLG is a CLG. (Hint: a graph of coherent groups over fg abelian groups is coherent.)*

Exercise 13. *Every abelian subgroup of a CLG Γ is fg and free, and there is a uniform bound to the rank. There is a finite $K(\Gamma, 1)$.*

Exercise 14. *Every non-abelian, freely indecomposable CLG admits a* principal splitting *over $\mathbb{Z}$: $A *_{\mathbb{Z}} B$ or $A *_{\mathbb{Z}}$ with A, B non-cyclic, and in the latter case $\mathbb{Z}$ is maximal abelian in the whole group.*

Exercise 15. *Every CLG is ω-residually free.*

The last exercise is more difficult than the others. It explains where the conditions in the definition of CLG come from. The idea is to construct homomorphisms $G \to \mathbb{F}$ by choosing complicated modular automorphisms of G, composing with ρ and then with a homomorphism to $\mathbb{F}$ that comes from the inductive assumption.

Example 1.29. Consider an index 2 subgroup H of an fg free group F and choose $g \in F \setminus H$. Suppose that $G := H *_{\langle g^2 \rangle} \langle g \rangle$ is freely indecomposable and admits no principal cyclic splitting. There is the obvious map $G \to F$, but G is not a limit group (Exercise 14 and Theorem 1.30). This shows the necessity of the last condition in the definition of CLG's. [13]

In Section 6, we will show:

Theorem 1.30. *For an fg group G, the following are equivalent.*

1. *G is a CLG.*

2. *G is ω-residually free.*

3. *G is a limit group.*

The fact that ω-residually free groups are CLG's is due to O. Kharlampovich and A. Myasnikov [16]. Limit groups act freely on $\mathbb{R}^n$-trees; see Remeslennikov [27] and Guirardel [13]. Kharlampovich-Myasnikov [15] prove that limit groups act freely on $\mathbb{Z}^n$-trees where $\mathbb{Z}^n$ is lexicographically ordered. Remeslennikov [26] also demonstrated that 2-residually free groups are ω-residually free.

[13] The element $g := a^2 b^2 a^{-2} b^{-1} \notin H := \langle a, b^2, bab^{-1} \rangle \subset F := \langle a, b \rangle$ is such an example. This can be seen from the fact that if $\langle x, y, z \rangle$ denotes the displayed basis for H, then $g^2 = x^2 y x^{-2} y^{-1} z^2 y z^{-2}$ is Whitehead reduced and each basis element occurs at least 3 times.