CHAPTER ONE

The Binary System

> A red ticket showed between wiper and windshield; I carefully tore it into two, four, eight pieces.
>
> – Vladimir Nabokov, *Lolita*

THE NUMBER SYSTEM now in use throughout the civilized world is a decimal system based on successive powers of 10. The digit at the extreme right of any number stands for a multiple of 10^0, or 1. The second digit from the right indicates a multiple of 10^1; the third digit, a multiple of 10^2; and so on. Thus 789 expresses the sum of $(7 \times 10^2) + (8 \times 10^1) + (9 \times 10^0)$. The widespread use of 10 as a number base is almost certainly due to the fact that we have 10 fingers; the very word "digit" reflects this. If a planet is inhabited by humanoids with 12 fingers, it is a good bet that their arithmetic uses a notation based on 12.

The simplest of all number systems that make use of the positions of digits is the binary system, based on the powers of 2. Some primitive tribes count in binary fashion, and ancient Chinese mathematicians knew about the system, but it was the great German mathematician Gottfried Wilhelm von Leibniz who seems to have been the first to develop the system in any detail. For Leibniz, it symbolized a deep metaphysical truth. He regarded 0 as an emblem of nonbeing or nothing; 1 as an emblem of being or substance. Both are necessary to the Creator, because a cosmos containing only pure substance would be indistinguishable from the empty cosmos, devoid of sound and fury and signified by 0. Just as in the binary system any integer can be expressed by a suitable placing of 0's and l's, so the mathematical structure of the entire created world becomes possible, Leibniz believed, as a consequence of the primordial binary split between being and nothingness.

From Leibniz's day until very recently the binary system was little more than a curiosity, of no practical value. Then came the computers! Wires either do or do not carry a current, a switch is on or off, a magnet is north–south or south–north, a flip-flop memory circuit is flipped or flopped. For such reasons enormous speed and accuracy are obtained by constructing computers that can process data coded in binary form. "Alas!" writes Tobias Dantzig in his book *Number, the Language of Science*, "what was once hailed as a monument to monotheism ended in the bowels of a robot."

Many mathematical recreations involve the binary system: the game of Nim, mechanical puzzles such as the Tower of Hanoi and the Rings of Cardan, and countless card tricks and brainteasers. Here we shall restrict our attention to a familiar set of "mind-reading" cards and a closely related set of punch cards with which several remarkable binary feats can be performed.

The construction of the mind-reading cards is made clear in Figure 1. On the left are the binary numbers from 0 through 31. Each digit in a binary number stands for a power of 2, beginning with 2^0 (or 1) at the extreme right, then proceeding leftward to 2^1 (or 2), 2^2, 2^3, and so on. These powers of 2 are shown at the top of the columns. To translate a binary number into its decimal equivalent, simply sum the powers of 2 that are expressed by the positions of the 1's. Thus 10101 represents $16 + 4 + 1$, or 21. To change 21 back to the binary form, a reverse procedure is followed. Divide 21 by 2. The result is 10 with a remainder of 1. This remainder is the first digit on the right of the binary number. Next divide 10 by 2. There is no remainder, so the next binary digit is 0. Then 5 is divided by 2, and so on until the binary number 10101 is completed. In the last step, 2 goes into 1 no times, with a remainder of 1.

The table of binary numbers is converted to a set of mind-reading cards simply by replacing each 1 with the decimal number that corresponds to the binary number in which the 1 occurs. The result is shown at the right side of the illustration. Each column of numbers is copied on a separate card. Hand the five cards to someone; ask him or her to think of any number from 0 to 31 inclusive and then to hand you all the cards on which his or her number appears. You can

BINARY NUMBERS / MIND-READING CARDS

	16	8	4	2	1					
0					0					
1					1					1
2				1	0				2	
3				1	1				3	3
4			1	0	0			4		
5			1	0	1			5		5
6			1	1	0			6	6	
7			1	1	1			7	7	7
8		1	0	0	0		8			
9		1	0	0	1		9			9
10		1	0	1	0		10		10	
11		1	0	1	1		11		11	11
12		1	1	0	0		12	12		
13		1	1	0	1		13	13		13
14		1	1	1	0		14	14	14	
15		1	1	1	1		15	15	15	15
16	1	0	0	0	0	16				
17	1	0	0	0	1	17				17
18	1	0	0	1	0	18			18	
19	1	0	0	1	1	19			19	19
20	1	0	1	0	0	20		20		
21	1	0	1	0	1	21		21		21
22	1	0	1	1	0	22		22	22	
23	1	0	1	1	1	23		23	23	23
24	1	1	0	0	0	24	24			
25	1	1	0	0	1	25	25			25
26	1	1	0	1	0	26	26		26	
27	1	1	0	1	1	27	27		27	27
28	1	1	1	0	0	28	28	28		
29	1	1	1	0	1	29	29	29		29
30	1	1	1	1	0	30	30	30	30	
31	1	1	1	1	1	31	31	31	31	31

Figure 1. Numbers on a set of mind-reading cards, shown on the right, are based on the binary numbers shown on the left. (Artist: Harold Jacobs)

immediately name the number. To learn it, you have only to add the top numbers of the cards given to you.

How does it work? Each number appears on a unique combination of cards, and this combination is equivalent to the binary notation for that number. When you total the top numbers on the cards, you are simply adding the powers of 2 that are indicated by the 1's in the binary version of the chosen number. The working of the trick can be further disguised by using cards of five different colors. You can then stand across the room and tell your subject to put all the cards bearing his or her number into a certain pocket and all remaining cards into another pocket. You, of course, must observe this, remembering which power of 2 goes with which color. Another presentation is to put the five (uncolored) cards in a row on the table. Stand across the room and ask the spectator to turn face down those cards that bear his or her number. Because you arranged the cards with their top numbers in order, you have only to observe which cards are reversed to know which key numbers to add.

The binary basis of punch-card sorting is amusingly dramatized by the set of cards depicted in Figure 2. They can be made easily from a set of 32 file cards. The holes should be a trifle larger than the diameter of a pencil. It is a good plan to cut five holes in one card and then use this card as a stencil for outlining holes on the other cards. If no punching device is available, the cutting of the holes with scissors can be speeded up by holding three cards as one and cutting them simultaneously. The cut-off corners make it easy to keep the cards properly oriented. After five holes have been made along the top of each card, the margin is trimmed away above certain holes as shown in the illustration. These notched holes correspond to the digit 1; the remaining holes correspond to 0. Each card carries in this way the equivalent of a binary number. The numbers run from 0 through 31, but in the illustration the cards are randomly arranged. Three unusual stunts can be performed with these cards. They may be troublesome to make, but everyone in the family will enjoy playing with them.

The first stunt consists of quickly sorting the cards so that their numbers are in serial order. Mix the cards any way you please and then square them like a deck of playing cards. Insert a pencil through hole E and lift up an inch or so. Half the cards will cling to the pencil, and half will be left behind. Give the pencil some vigorous shakes to make sure all cards drop that are supposed to drop, and then raise the pencil higher until the cards are separated into two halves. Slide the packet off the pencil and put it in *front* of the other cards. Repeat this procedure with each of the other holes,

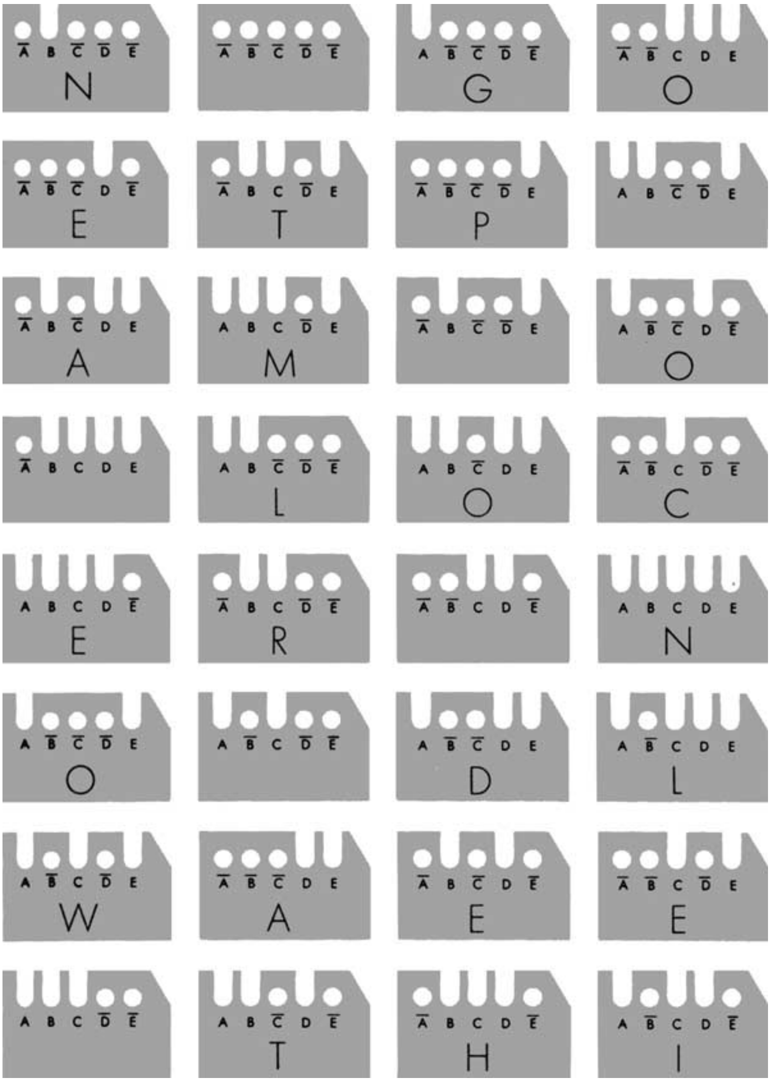

Figure 2. A set of punch cards that will unscramble a message, guess a selected number, and solve logic problems. (Artist: Patra McElwee)

taking them right to left. After the fifth sorting, it may surprise you to find that the binary numbers are now in serial order, beginning with 0 on the card facing you. Flip through the cards and read the Christmas message printed on them!

The second stunt uses the cards as a computer for determining the selected number on the set of mind-reading cards. Begin with the punch cards in any order. Insert the pencil in hole E and ask if the chosen number appears on the card with a top number of 1. If the reply is yes, then lift up on the pencil and discard all cards that cling to it. If the reply is no, then discard all cards left behind. You now have a packet of 16 cards. Ask if the number is on the card with a top number of 2, and then repeat the procedure with the pencil in hole D. Continue in this manner with the remaining cards and holes. You will end with a single punch card, and its binary number will be the chosen number. If you like, print decimal numbers on all the cards so that you will not have to translate the binary numbers.

The third stunt employs the cards as a logic computer in a manner first proposed by William Stanley Jevons, the English economist and logician. Jevons' "logical abacus," as he called it, used flat pieces of wood with steel pins at the back so that they could be lifted from a ledge, but the punch cards operate in exactly the same way and are much simpler to make. Jevons also invented a complex mechanical device, called the "logic piano," which operated on the same principles, but the punch cards will do all that his piano could do. In fact, they will do more, because the piano took care of only four terms, and the cards take care of five.

The five terms A, B, C, D, and E are represented by the five holes, which in turn represent binary digits. Each 1 (or notched hole) corresponds to a true term; each 0, to a false term. A line over the top of a letter indicates that the term is false; otherwise it is true. Each card is a unique combination of true and false terms, and because the 32 cards exhaust all possible combinations, they are the equivalent of what is called a "truth table" for the five terms. The operation of the cards is best explained by showing how they can be used for solving a problem in two-valued logic.

The following puzzle appears in *More Problematical Recreations*, a booklet issued by Litton Industries, founded by Charles "Tex" Thornton in 1953 in Beverly Hills, California. "If Sara shouldn't, then Wanda would. It is impossible that the statements: 'Sara should,' and 'Camille couldn't,' can both be true at the same time. If Wanda would, then Sara should and Camille could. Therefore Camille could. Is the conclusion valid?"

To solve this problem, start with the punch cards in any order. Only three terms are involved, so we shall be concerned with only the A, B, and C holes.

$$A = \text{Sara should}$$
$$\bar{A} = \text{Sara shouldn't}$$
$$B = \text{Wanda would}$$
$$\bar{B} = \text{Wanda wouldn't}$$
$$C = \text{Camille could}$$
$$\bar{C} = \text{Camille couldn't}$$

The problem has three premises. The first – "If Sara shouldn't, then Wanda would" – tells us that the combination of $\bar{A}$ and $\bar{B}$ is not permitted, so we must eliminate all cards bearing this combination. It is done as follows. Insert the pencil in A and lift. All cards on the pencil bear $\bar{A}$. Hold them as a group, remove the pencil, put it in B, and lift again. The pencil will raise all cards bearing both $\bar{A}$ and $\bar{B}$, which is the invalid combination, so these cards are discarded. All remaining cards are assembled into a pack once more (the order does not matter), and we are ready for the second premise.

The second premise is that "Sara should" and "Camille couldn't" cannot both be true. In other words, we cannot permit the combination $A\bar{C}$. Insert the pencil in A and lift up all cards bearing $\bar{A}$. These are *not* the cards we want, so we place them temporarily aside and continue with the A group that remains. Insert the pencil in C and raise the $\bar{C}$ cards. These bear the invalid combination $A\bar{C}$, so they are permanently discarded. Assemble the remaining cards once more.

The last premise tells us that if Wanda would, then Sara should and Camille could. A bit of reflection will show that this eliminates two combinations: $\bar{A}B$ and $B\bar{C}$. Put the pencil in hole A, lift, and continue working with the lifted cards. Insert the pencil in B; lift. No cards cling to the pencil. This means that the two previous premises have already eliminated the combination $\overline{AB}$. Because the cards all bear $\bar{A}B$ (an invalid combination), this entire packet is permanently discarded. The only remaining task is to eliminate $B\bar{C}$ from the remaining cards. The pencil in B will lift out the $\bar{B}$ cards, which are placed temporarily aside. When the pencil is put in C of the cards that remain, no cards can be lifted, indicating that the invalid combination of $B\overline{C}$ has already been ruled out by previous steps.

We are thus left with eight cards, each bearing a combination of truth values for A, B, and C that is consistent with all three premises. These combinations are the valid lines of a truth table for the combined premises. An inspection of the cards reveals that C is true on all eight, so it is correct to conclude that Camille could. Other conclusions can also be deduced from the premises. We can, for example, assert that Sara should. But the interesting question of whether Wanda would or wouldn't remains, at least in the light of available knowledge, an inscrutable binary mystery.

For those who would like another problem to feed the cards, here is an easy one. In a suburban home live Abner, his wife Beryl, and their three children, Cleo, Dale, and Ellsworth. The time is 8 P.M. on a winter evening.

1. If Abner is watching television, so is his wife.
2. Either Dale or Ellsworth, or both of them, are watching television.
3. Either Beryl or Cleo, but not both, is watching television.
4. Dale and Cleo are either both watching or both not watching television.
5. If Ellsworth is watching television, then Abner and Dale are also watching.

Who is watching television and who is not?

ADDENDUM

Edward B. Grossman, New York City, wrote to say that a variety of commercial cards for binary filing and sorting are now available in large stationery supply stores. Holes are preperforated and one can buy special punches for making the slots. The holes are too small to take pencils, but one can use knitting needles, Q-Tip sticks, opened-out paper clips, or the sorting rods that come with some makes of cards.

Giuseppe Aprile, a professor of engineering at the University of Palermo, Italy, sent the two photographs shown in Figure 3. Quick, errorless separation of the cards is made possible by a complementary row of holes and notches at the bottom edge of each card.

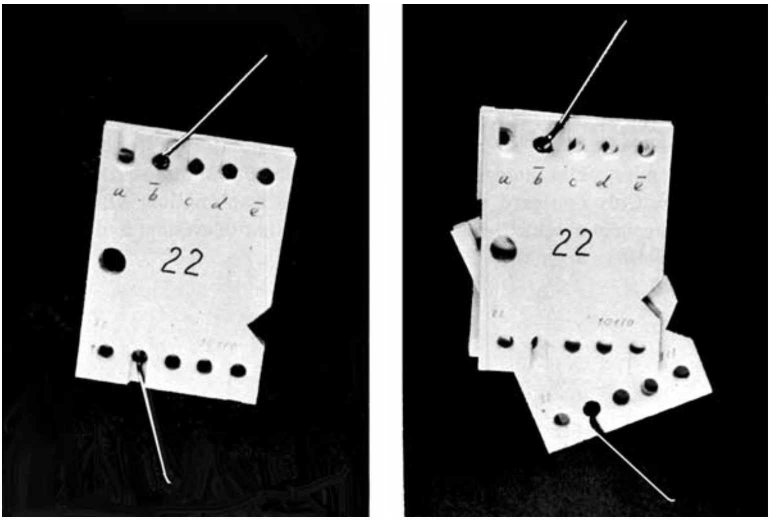

Figure 3. A complementary row of holes at the bottom of cards permits errorless sorting.

Pins through complementary holes in the bottom row anchor the set of cards that remains when pins through the top holes remove a set of cards.

ANSWERS

The logic problem can be solved with the punch cards as follows: Let A, B, C, D, and E stand for Abner, Beryl, Cleo, Dale, and Ellsworth. A term is true if the person is watching television; otherwise it is false. Premise 1 eliminates all cards bearing $A\overline{B}$; premise 2 eliminates $\overline{D}E$; premise 3 eliminates BC and $\overline{B}\overline{C}$; premise 4 eliminates $\overline{C}D$ and $C\overline{D}$; premise 5 eliminates $\overline{A}E$ and $\overline{D}E$. Only one card remains, bearing the combination $\overline{A}\overline{B}CD\overline{E}$. We conclude that Cleo and Dale are watching television, and that the others are not.

POSTSCRIPT

Paul Swinford, a semiprofessional magician in Cincinnati, devised a deck of playing cards called the Cyberdeck that has holes and slots along the top and bottom edges of the cards. A variety of bewildering tricks can be performed with this deck, as explained by Swinford in his booklet *The Cyberdeck* (1986). Both cards and booklet are distributed through magic supply stores.

The following anonymous statement may take a while before you fully understand it: There are 10 types of people – those who understand binary notation and those who don't.

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