CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Lectures in Logic and Set Theory Volume 1

This two-volume work bridges the gap between introductory expositions of logic or set theory on one hand, and the research literature on the other. It can be used as a text in an advanced undergraduate or beginning graduate course in mathematics, computer science, or philosophy. The volumes are written in a user-friendly conversational lecture style that makes them equally effective for self-study or class use.

Volume 1 includes formal proof techniques, a section on applications of compactness (including non-standard analysis), a generous dose of computability and its relation to the incompleteness phenomenon, and the first presentation of a complete proof of Gödel's second incompleteness theorem since Hilbert and Bernay's *Grundlagen*.

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LECTURES IN LOGIC AND SET THEORY

Volume 1: Mathematical Logic

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Contents

Preface

Both volumes in this series are about what mathematicians, especially logicians, call the "foundations" (of mathematics) – that is, the tools of the axiomatic method, an assessment of their effectiveness, and two major examples of application of these tools, namely, in the development of number theory and set theory.

There have been, in hindsight, two main reasons for writing this volume. One was the existence of notes I wrote for my lectures in mathematical logic and computability that had been accumulating over the span of several years and badly needed sorting out. The other was the need to write a small section on logic, "A Bit of Logic" as I originally called it, that would bootstrap my volume on set theory† on which I had been labouring for a while. Well, one thing led to another, and a 30 or so page section that I initially wrote for the latter purpose grew to become a self-standing volume of some 300 pages. You see, this material on logic is a good story and, as with all good stories, one does get carried away wanting to tell more.

I decided to include what many people will consider, I should hope, as being the absolutely essential topics in *proof, model*, and *recursion* theory – "absolutely essential" in the context of courses taught near the upper end of undergraduate, and at the lower end of graduate curricula in mathematics, computer science, or philosophy. But no more.‡ This is the substance of Chapter I; hence its title "Basic Logic".

† A chapter by that name now carries out these bootstrapping duties – the proverbial "Chapter 0" (actually Chapter I) of volume 2.

‡ These topics include the foundation and development of non-standard analysis up to the extreme value theorem, elementary equivalence, diagrams, and Löwenheim-Skolem theorems, and Gödel's first incompleteness theorem (along with Rosser's sharpening).

But then it occurred to me to also say something about one of the most remarkable theorems of logic – arguably *the* most remarkable – about the limitations of formalized theories: Gödel's second incompleteness theorem. Now, like most reasonable people, I never doubted that this theorem is true, but, as the devil is in the details, I decided to learn its proof – right from Peano's axioms. What better way to do this than writing down the proof, gory details and all? This is what Chapter II is about.[†]

As a side effect, the chapter includes many theorems and techniques of one of the two most important – from the point of view of foundations – "applied" logics (formalized theories), namely, Peano arithmetic (the other one, set theory, taking all of volume 2).

I have hinted above that this (and the second) volume are aimed at a fairly advanced reader: The level of exposition is designed to fit a spectrum of mathematical sophistication from third year undergraduate to junior graduate level (each group will find here its favourite sections that serve its interests and level of preparation – and should not hesitate to judiciously omit topics).

There are no specific prerequisites beyond some immersion in the "proof culture", as this is attainable through junior level courses in calculus, linear algebra, or discrete mathematics. However, some familiarity with concepts from elementary naïve set theory such as finiteness, infinity, countability, and uncountability will be an asset.[‡]

A word on approach. I have tried to make these lectures user-friendly, and thus accessible to readers who do not have the benefit of an instructor's guidance. Devices to that end include anticipation of questions, frequent promptings for the reader to rethink an issue that might be misunderstood if glossed over ("Pauses"), and the marking of important passages, by ☡, as well as those that can be skipped at first reading, by ☡ ☡.

Moreover, I give (mostly) *very* detailed proofs, as I know from experience that omitting details normally annoys students.

[†] It is strongly conjectured here that this is the only complete proof in print other than the one that was given in Hilbert and Bernays (1968). It is fair to clarify that I use the term "complete proof" with a strong assumption in mind: That the axiom system we start with is *just* Peano arithmetic. Proofs based on a stronger – thus technically more convenient – system, namely, *primitive recursive arithmetic*, have already appeared in print (Diller (1976), Smoryński (1985)). The difficulty with using Peano arithmetic as the starting point is that the only primitive recursive functions initially available are the successor, identity, plus, and times. An awful amount of work is needed – a preliminary "coding trick" – to prove that all the rest of the primitive recursive functions "exist". By then are we already midway in Chapter II, and only then are we ready to build Gödel numbers of terms, formulas, and proofs and to prove the theorem.

[‡] I have included a short paragraph nicknamed "a crash course on countable sets" (Section I.5, p. 62), which certainly helps. But having seen these topics before helps even more.

The first chapter has a lot of exercises (the second having proportionally fewer). Many of these have hints, but none are marked as "hard" vs. "just about right", a subjective distinction I prefer to avoid. In this connection here is some good advice I received when I was a graduate student at the University of Toronto: "Attempt all the problems. Those you can do, don't do. Do the ones you cannot".

What to read. Consistently with the advice above, I suggest that you read this volume from cover to cover – including footnotes! – skipping only what you already know. Now, in a class environment this advice may be impossible to take, due to scope and time constraints. An undergraduate (one semester) course in logic at the third year level will probably cover Sections I.1–I.5, making light of Section I.2, and will introduce the student to the elements of computability along with a hand-waving "proof" of Gödel's first incompleteness theorem (the "semantic version" ought to suffice). A fourth year class will probably attempt to cover the entire Chapter I. A first year graduate class has no more time than the others at its disposal, but it usually goes much faster, skipping over familiar ground, thus it will probably additionally cover Peano arithmetic and will get to see how Gödel's second theorem follows from Löb's derivability conditions.

Acknowledgments. I wish to offer my gratitude to all those who taught me, a group led by my parents and too large to enumerate. I certainly include my students here. I also include Raymond Wilder's book on the foundations of mathematics, which introduced me, long long ago, to this very exciting field and whetted my appetite for more (Wilder (1963)).

I should like to thank the staff at Cambridge University Press for their professionalism, support, and cooperation, with special appreciation due to Lauren Cowles and Caitlin Doggart, who made all the steps of this process, from refereeing to production, totally painless.

This volume is the last installment of a long project that would have not been successful without the support and warmth of an understanding family (thank you).

I finally wish to record my appreciation to Donald Knuth and Leslie Lamport for the typesetting tools TEX and LATEX that they have made available to the technical writing community, making the writing of books such as this one almost easy.

George Tourlakis
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