SOLITON EQUATIONS AND THEIR ALGEBRO-GEOMETRIC SOLUTIONS

Volume I: (1 + 1)-Dimensional Continuous Models

FRITZ GESZTESY
University of Missouri
Columbia, Missouri
USA

HELGE HOLDEN
Norwegian University of
Science and Technology
Trondheim, Norway

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times 10/13 pt. *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data
Gesztesy, Fritz, 1953–
Soliton equations and their algebro-geometric solutions / Fritz Gesztesy,
Helge Holden.
p. cm. – (Cambridge studies in advanced mathematics ; 79)
Includes bibliographical references and index.
Contents: v. 1. (1 + 1)-dimensional continuous models
ISBN 0-521-75307-4 (v. 1)
1. Differential equations, Nonlinear – Numerical solutions. 2. Solitons.
I. Holden, H. (Helge), 1956– II. Title. III. Series.
QC20.7.D5 G47 2003
530.15′5355 – dc21 2002074069

ISBN 0 521 75307 4 hardback

Contents

Introduction

> It often happens that the understanding of the mathematical nature of an equation is impossible without a detailed understanding of its solutions.
>
> *Freeman J. Dyson*

Background: The discovery of solitary waves of translation goes back to Scott Russell in 1834, and during the remaining part of the 19th century the true nature of these waves remained controversial. It was only with the derivation by Korteweg and de Vries in 1895 of what is now called the Korteweg–de Vries (KdV) equation, that the one-soliton solution and hence the concept of solitary waves was put on a firm basis.[1] An extraordinary series of events took place around 1965 when Kruskal and Zabusky, while analyzing the numerical results of Fermi, Pasta, and Ulam on heat conductivity in solids, discovered that pulselike solitary wave solutions of the KdV equation, for which the name "solitons" was coined, interact elastically. This was followed by the 1967 discovery of Gardner, Greene, Kruskal, and Miura that the inverse scattering method allows one to solve initial value problems for the KdV equation with sufficiently fast-decaying initial data. Soon thereafter, in 1968, Lax found a new explanation of the isospectral nature of KdV solutions using the concept of Lax pairs and introduced a whole hierarchy of KdV equations. Subsequently, in the early 1970s, Zakharov and Shabat (ZS), and Ablowitz, Kaup, Newell, and Segur (AKNS) extended the inverse scattering method to a wide class of nonlinear partial differential equations of relevance in various scientific contexts ranging from nonlinear optics to condensed matter physics and elementary particle physics. In particular, solitons found numerous applications in classical and quantum field theory and in connection with optical communication devices.

Another decisive step forward in the development of completely integrable soliton equations was taken around 1974. Prior to that period, inverse spectral

[1] With hindsight, though, it is now clear that other researchers, such as Boussinesq, derived the KdV equation and its one-soliton solution prior to 1895, as described in the notes to Section 1.1.

methods in the context of nonlinear evolution equations had been restricted to spatially decaying solutions. In 1974–75, the arsenal of inverse spectral methods was extended considerably in scope to include periodic and certain classes of quasi-periodic and almost periodic KdV solutions. This new approach to constructing solutions of integrable nonlinear evolution equations, partly based on inverse spectral theory and partly relying on algebro-geometric methods developed by pioneers such as Dubrovin, Flaschka, Its, Krichever, Lax, Marchenko, Matveev, McKean, Novikov, van Moerbeke – to name just a few – was followed by very rapid development in the field. Within a few years of intense activity worldwide, the landscape of integrable systems was changed forever. By the early 1980s the theory was extended to a large class of nonlinear (including some multi-dimensional) evolution equations beyond the KdV equation, and the explicit theta function representations of quasi-periodic solutions of integrable equations (including, e.g., soliton solutions as special limiting cases) had introduced new algebro-geometric techniques into this area of nonlinear partial differential equations. Subsequently, this led to several new and deep results in nonlinear partial differential equations as well as in algebraic geometry (such as a solution of Schottky's problem).

Our series of monographs is devoted to this area of algebro-geometric solutions of hierarchies of soliton equations.

Scope: We aim for an elementary, yet self-contained and precise, presentation of hierarchies of integrable soliton equations and their algebro-geometric solutions. Our point of view is predominantly influenced by analytical methods, especially by spectral theoretic techniques. We hope this will make the presentation accessible and attractive to analysts working outside the traditional areas associated with soliton equations. Central to our approach is a simultaneous construction of all algebro-geometric solutions and their theta function representation of a given hierarchy. In this volume we focus on some of the key hierarchies in $(1+1)$-dimensions associated with continuous integrable models such as the Korteweg–de Vries hierarchy (KdV), the combined sine–Gordon modified Korteweg–de Vries hierarchy (sGmKdV), the Ablowitz–Kaup–Newell–Segur hierarchy[1] (AKNS), the classical massive Thirring system (Th), and the Camassa–Holm hierarchy (CH). The key equations defining the corresponding hierarchies read

$$\begin{aligned}
&\text{KdV:} && u_t + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x = 0,\\
&\text{sGmKdV:} && u_{xt} - \sin(u) = 0,\\
&\text{AKNS:} && \begin{pmatrix} p_t + \frac{i}{2}p_{xx} - ip^2q \\ q_t - \frac{i}{2}q_{xx} + ipq^2 \end{pmatrix} = 0,
\end{aligned} \tag{0.1}$$

[1] Using the gauge equivalence of the AKNS hierarchy and classical Boussinesq hierarchy, we also treat the latter.

$$\text{Th:} \qquad \begin{pmatrix} -iu_x + 2v + 2vv^*u \\ iu_x^* + 2v^* + 2vv^*u^* \\ -iv_t + 2u + 2uu^*v \\ iv_t^* + 2u^* + 2uu^*v^* \end{pmatrix} = 0,$$

$$\text{CH:} \qquad 4u_t - u_{xxt} - 2uu_{xxx} - 4u_xu_{xx} + 24uu_x = 0.$$

Our principal goal in this monograph is the construction of algebro-geometric solutions of the hierarchies associated with the equations listed in (0.1). Interest in the class of algebro-geometric solutions can be motivated in a variety of ways: It represents a natural extension of the classes of soliton and rational solutions, and similar to these, its elements can still be regarded as explicit solutions of the nonlinear integrable evolution equation in question (even though their complexity considerably increases compared with soliton solutions due to the underlying analysis on compact Riemann surfaces). Moreover, algebro-geometric solutions can be used to approximate more general solutions (such as almost periodic ones), although this is not a topic pursued in this monograph. Here we primarily focus on the construction of explicit solutions in terms of certain algebro-geometric data on a compact Riemann surface and their representation in terms of theta functions. For instance, in KdV-type contexts, solitons arise as the special case of solutions corresponding to an underlying singular hyperelliptic curve obtained by confluence of two or more branch points, and rational solutions correspond to a further singularization of the original curve. In either case, the theta function associated with the underlying algebraic curve degenerates into appropriate determinants with exponential, respectively, rational entries.

We use basic techniques from the theory of differential equations, some spectral analysis, and elements of algebraic geometry (most notably, the basic theory of compact Riemann surfaces). In particular, we do not employ more advanced tools such as loop groups, Grassmanians, Lie algebraic considerations, formal pseudo-differential expressions, etc. However, occasionally we bridge the gap to spectral theory and its vicinity and include some finer points of the basic formalism often omitted in this context. Thus, this volume strays off the mainstream, but we hope it appeals to spectral theorists and their kin and convinces them of the beauty of the subject. In particular, we hope a reader interested in quickly penetrating to the fundamentals of the algebro-geometric approach of constructing solutions of hierarchies of completely integrable evolution equations will not be disappointed.

Completely integrable systems, and especially nonlinear evolution equations of soliton-type, are an integral part of modern mathematical and theoretical physics with far-reaching implications from pure mathematics to the applied sciences. We intend to contribute to the dissemination of some of the beautiful techniques applied in this area.

Contents: In the present volume we provide an effective approach to the construction of algebro-geometric solutions of certain completely integrable nonlinear evolution equations by developing a technique that simultaneously applies to all equations of the hierarchy in question.

Starting with a specific integrable partial differential equation, one can build an infinite sequence of higher-order partial differential equations, the so-called hierarchy of the original soliton equation, by developing an explicit recursive formalism that reduces the construction of the entire hierarchy to elementary manipulations with polynomials and defines the associated Lax pairs or zero-curvature equations. Using this recursive polynomial formalism, we simultaneously construct algebro-geometric solutions for the entire hierarchy of soliton equations at hand. On a more technical level, our point of departure for the construction of algebro-geometric solutions is not directly based on Baker–Akhiezer functions and axiomatizations of algebro-geometric data but rather on Dubrovin-type equations, trace formulas, and a canonical meromorphic function ϕ on the underlying hyperelliptic Riemann surface $\mathcal{K}_n$ of genus $n \in \mathbb{N}$. More precisely, this fundamental meromorphic function ϕ carries the spectral information of the underlying Lax operator (such as the Schrödinger and Dirac operators in the KdV and AKNS contexts) and in many instances represents a direct generalization of the Weyl–Titchmarsh m-function, a fundamental device in the spectral theory of ordinary differential operators. Riccati-type differential equations satisfied by ϕ separately in the space and time variables then govern the time evolutions of all quantities of interest (such as that of the associated Baker–Akhiezer vector). The basic meromorphic function ϕ on $\mathcal{K}_n$ is then linked with solutions of equations of the underlying hierarchy via trace formulas and Dubrovin-type equations for (projections of) the pole divisor of ϕ. Subsequently, the Riemann theta function representation of ϕ is then obtained more or less simultaneously with those of the Baker–Akhiezer vector and the algebro-geometric solutions of the (stationary or time-dependent) equations of the hierarchy of evolution equations. This concisely summarizes our approach to all the $(1+1)$-dimensional, continuous integrable models discussed in this volume.

In the following we will detail this verbal description of our approach to algebro-geometric solutions of integrable hierarchies with the help of the KdV hierarchy. The latter consists of a sequence of nonlinear evolution equations for a function $u = u(x, t)$, the most prominent element of which, the KdV equation itself, is given by

$$u_t + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x = 0. \tag{0.2}$$

The KdV hierarchy is the simplest of all the hierarchies of nonlinear evolution equations studied in this volume, but the same strategy, with modifications to be discussed in the individual chapters, applies to all integrable systems treated in this monograph and is in fact typical for all $(1+1)$-dimensional integrable hierarchies of soliton equations.

A discussion of the KdV case then proceeds as follows.[1] In order to define the Lax pairs and zero-curvature pairs for the KdV hierarchy, one assumes u to be a smooth function on $\mathbb{R}$ (or meromorphic in $\mathbb{C}$) in the stationary context or a smooth function on $\mathbb{R}^2$ in the time-dependent case, and one introduces the recursion relation for some functions f_ℓ of u by

$$f_0 = 1, \quad f_{\ell,x} = -(1/4)f_{\ell-1,xxx} + uf_{\ell-1,x} + (1/2)u_x f_{\ell-1}, \quad \ell \in \mathbb{N}. \quad (0.3)$$

Given the recursively defined sequence $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ (whose elements turn out to be differential polynomials with respect to u defined up to certain integration constants) one defines the Lax pair of the KdV hierarchy by

$$L = -\frac{d^2}{dx^2} + u, \quad (0.4)$$

$$P_{2n+1} = \sum_{\ell=0}^{n} \Big(f_{n-\ell}\frac{d}{dx} - \frac{1}{2}f_{n-\ell,x}\Big)L^\ell. \quad (0.5)$$

The commutator of P_{2n+1} and L then reads[2]

$$[P_{2n+1}, L] = 2f_{n+1,x}, \quad (0.6)$$

using the recursion (0.3). Introducing a deformation (time) parameter[3] $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$ into u, the *KdV hierarchy* of nonlinear evolution equations is then defined by imposing the *Lax commutator relations*

$$\frac{d}{dt_n}L - [P_{2n+1}, L] = 0, \quad (0.7)$$

for each $n \in \mathbb{N}_0$. By (0.6), the latter are equivalent to the collection of evolution equations[4]

$$\mathrm{KdV}_n(u) = u_{t_n} - 2f_{n+1,x}(u) = 0, \quad n \in \mathbb{N}_0. \quad (0.8)$$

Explicitly,

$$\begin{aligned}
\mathrm{KdV}_0(u) &= u_{t_0} - u_x = 0,\\
\mathrm{KdV}_1(u) &= u_{t_1} + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x - c_1u_x = 0,\\
\mathrm{KdV}_2(u) &= u_{t_2} - \tfrac{1}{16}u_{xxxxx} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_xu_{xx} - \tfrac{15}{8}u^2u_x\\
&\quad + c_1\big(\tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x\big) - c_2u_x = 0, \quad \text{etc.,}
\end{aligned}$$

[1] All details of the following construction are to be found in Chapter 1.
[2] The quantities P_{2n+1} and $\{f_\ell\}_{\ell=0,\dots,n}$ are constructed in such a manner that all differential operators in the commutator (0.6) vanish.
[3] Here we follow Hirota's notation and introduce a separate time variable t_n for the nth level in the KdV hierarchy.
[4] In a slight abuse of notation, we will occasionally stress the functional dependence of f_ℓ on u, writing $f_\ell(u)$.

represent the first few equations of the time-dependent KdV hierarchy. For $n = 1$ and $c_1 = 0$, we obtain *the* KdV equation (0.2). Introducing the polynomials ($z \in \mathbb{C}$),

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^{\ell}, \tag{0.9}$$

$$G_{n-1}(z) = -F_{n,x}(z)/2, \tag{0.10}$$

$$H_{n+1}(z) = (z - u)F_n(z) + (1/2)F_{n,xx}(z), \tag{0.11}$$

one can alternatively introduce the KdV hierarchy as follows. One defines a pair of 2×2 matrices $(U(z), V_{n+1}(z))$ depending polynomially on z by

$$U(z) = \begin{pmatrix} 0 & 1 \\ -z + u & 0 \end{pmatrix}, \tag{0.12}$$

$$V_{n+1}(z) = \begin{pmatrix} G_{n-1}(z) & F_n(z) \\ -H_{n+1}(z) & -G_{n-1}(z) \end{pmatrix}, \tag{0.13}$$

and then postulates the *zero-curvature equation*[1]

$$U_{t_n} - V_{n+1,x} + [U, V_{n+1}] = 0. \tag{0.14}$$

One easily verifies that both the Lax approach (0.8) as well as the zero-curvature approach (0.14) reduce to the basic equation

$$u_{t_n} + (1/2)F_{n,xxx} - 2(u - z)F_{n,x} - u_x F_n = 0. \tag{0.15}$$

Each one of (0.8), (0.14), and (0.15) defines the KdV hierarchy by varying $n \in \mathbb{N}_0$.

The strategy is as follows: We temporarily assume existence of a solution u and derive several of its properties. In particular, we show that u satisfies a trace formula (cf. (0.37) in the stationary case and (0.54) in the time-dependent case) expressed in terms of certain Dirichlet data that satisfy the so-called Dubrovin equations (cf. (0.38) in the stationary case and (0.55) in the time-dependent case), a first-order system of ordinary differential equations that can be shown at least locally to possess solutions. Furthermore, we deduce explicit formulas for the solution u, the so-called Its–Matveev formulas (cf. (0.40) in the stationary case and (0.57) in the time-dependent case).

The Lax and zero-curvature equations (0.7) and (0.14) imply a most remarkable isospectral deformation of L, as will be discussed later in this introduction. At this

[1] Equations $\Psi_x = U\Psi$, $\Psi_{t_n} = V_{n+1}\Psi$ and their compatibility condition (0.14), $U_{t_n} - V_{n+1,x} + [U, V_{n+1}] = 0$ permit a geometrical interpretation as follows: U and V_{n+1} may be considered local connection coefficients in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$ with space-time $\mathbb{R}^2$ the base and Ψ taking values in the fiber $\mathbb{C}^2$. The compatibility equation (0.14) then shows that the (U, V_{n+1})-connection has zero-curvature, and hence (0.14) is called a zero-curvature representation of a nonlinear evolution equation.

point, however, we interrupt our time-dependent KdV considerations for a while and take a closer look at the special stationary KdV equations defined by

$$u_{t_n} = 0, \quad n \in \mathbb{N}_0. \tag{0.16}$$

By (0.6)–(0.8) and (0.14), (0.15), the condition (0.16) is then equivalent to each one of the following collection of equations, with n ranging in $\mathbb{N}_0$, which then defines the *stationary KdV hierarchy*,

$$[P_{2n+1}, L] = 0, \tag{0.17}$$

$$f_{n+1,x} = 0, \tag{0.18}$$

$$-V_{n+1,x} + [U, V_{n+1}] = 0, \tag{0.19}$$

$$(1/2)F_{n,xxx} - 2(u - z)F_{n,x} - u_x F_n = 0. \tag{0.20}$$

To set the stationary KdV hierarchy apart from the general time-dependent one, we will denote it by

$$\text{s-KdV}_n(u) = -2f_{n+1,x}(u) = 0, \quad n \in \mathbb{N}_0.$$

Explicitly, the first few equations of the stationary KdV hierarchy then read as follows

$$\begin{aligned}
\text{s-KdV}_0(u) &= -u_x = 0, \\
\text{s-KdV}_1(u) &= \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x - c_1 u_x = 0, \\
\text{s-KdV}_2(u) &= -\tfrac{1}{16}u_{xxxxx} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_x u_{xx} - \tfrac{15}{8}u^2 u_x \\
&\quad + c_1\left(\tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x\right) - c_2 u_x = 0, \quad \text{etc.}
\end{aligned}$$

The class of *algebro-geometric* KdV potentials, by definition, equals the set of solutions u of the stationary KdV hierarchy. In the following analysis we fix the value of n in (0.17)–(0.20), and hence we now turn to the investigation of algebro-geometric solutions u of the nth equation within the stationary KdV hierarchy. Equation (0.17) is of special interest because, by a 1923 result of Burchnall and Chaundy, commuting differential expressions (due to a common eigenfunction to be discussed below, cf. (0.33), (0.34)) give rise to an algebraic relationship between the two differential expressions. Similarly, (0.19) permits the important conclusion that

$$\partial_x \det(yI_2 - iV_{n+1}(z, x)) = 0 \tag{0.21}$$

and hence

$$\begin{aligned}
\det(yI_2 - iV_{n+1}(z, x)) &= y^2 - \det(V_{n+1}(z, x)) \\
&= y^2 + G_{n-1}(z, x)^2 - F_n(z, x)H_{n+1}(z, x) = y^2 - R_{2n+1}(z)
\end{aligned} \tag{0.22}$$

for some x-independent monic polynomial R_{2n+1}, which we write as

$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m) \text{ for some } \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}.$$

In particular, the combination

$$F_n(z,x)H_{n+1}(z,x) - G_{n-1}(z,x)^2 = R_{2n+1}(z) \tag{0.23}$$

is x-independent. Moreover, (0.20) can easily be integrated to yield

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2 = R_{2n+1} \tag{0.24}$$

with precisely the same integration constant $R_{2n+1}(z)$ as in (0.22). In fact, by (0.10) and (0.11), equations (0.23) and (0.24) are simply identical. Incidentally, the algebraic relationship between L and P_{2n+1} alluded to in connection with the vanishing of their commutator in (0.17) can be made precise as follows: Restricting P_{2n+1} to the (algebraic) kernel $\ker(L-z)$ of $L-z$, one computes, using (0.5) and (0.24),

$$\begin{aligned}\left(P_{2n+1}\big|_{\ker(L-z)}\right)^2 &= -\left(\frac{1}{2}F_{n,xx}F_n - \frac{1}{4}F_{n,x}^2 - (u-z)F_n^2\right)\bigg|_{\ker(L-z)}\\ &= -R_{2n+1}(L)\big|_{\ker(L-z)}.\end{aligned}$$

Thus, one concludes that P_{2n+1}^2 and $-R_{2n+1}(L)$ coincide on $\ker(L-z)$, and since $z \in \mathbb{C}$ is arbitrary, one infers that

$$P_{2n+1}^2 + R_{2n+1}(L) = 0 \tag{0.25}$$

holds once again with the same polynomial R_{2n+1}. The characteristic equation of iV_{n+1} (cf. (0.22)) and (0.25) naturally lead one to the introduction of the *hyperelliptic curve* $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) defined by

$$\mathcal{K}_n \colon \mathcal{F}_n(z,y) = y^2 - R_{2n+1}(z) = 0, \quad R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m). \tag{0.26}$$

We compactify the curve by adding the point P_∞ (still denoting it by $\mathcal{K}_n$ for simplicity) and note that points P on the curve are denoted by $P = (z,y) \in \mathcal{K}_n \setminus \{P_\infty\}$, where $y(\,\cdot\,)$ is a meromorphic function on $\mathcal{K}_n$ satisfying[1] $y^2 - R_{2n+1}(z) = 0$. For simplicity, we will assume in the following that the (affine part of the) curve $\mathcal{K}_n$ is nonsingular, that is, the zeros E_m of R_{2n+1} are all simple. Remaining within the stationary framework a bit longer, one can now introduce the fundamental meromorphic function ϕ on $\mathcal{K}_n$ alluded to earlier as follows,

$$\phi(P,x) = \frac{iy - G_{n-1,x}(z,x)}{F_n(z,x)} \tag{0.27}$$

$$= \frac{-H_{n+1}(z,x)}{iy + G_{n-1,x}(z,x)}, \quad P = (z,y) \in \mathcal{K}_n. \tag{0.28}$$

[1] For more details, refer to Appendix B and Chapter 1.

Equality of the two expressions (0.27) and (0.28) is an immediate consequence of the identity (0.23) and the fact $y^2 = R_{2n+1}(z)$. A comparison with (0.19) then readily reveals that ϕ satisfies the Riccati-type equation

$$\phi_x + \phi^2 = u - z. \tag{0.29}$$

The next step is crucial. It concerns the zeros and poles of ϕ and hence involves the zeros of $F_n(\,\cdot\,, x)$ and $H_{n+1}(\,\cdot\,, x)$. Isolating the latter by introducing the factorizations

$$F_n(z,x) = \prod_{j=1}^{n}(z - \mu_j(x)), \quad H_{n+1}(z,x) = \prod_{\ell=0}^{n}(z - \nu_\ell(x)),$$

one can use the zeros of F_n and H_{n+1} to define the following points $\hat{\mu}_j(x)$, $\hat{\nu}_\ell(x)$ on $\mathcal{K}_n$,

$$\hat{\mu}_j(x) = (\mu_j(x), iG_{n-1,x}(\mu_j(x), x)), \quad j = 1, \dots, n, \tag{0.30}$$

$$\hat{\nu}_\ell(x) = (\nu_\ell(x), -iG_{n-1,x}(\nu_\ell(x), x)), \quad \ell = 0, \dots, n. \tag{0.31}$$

The motivation for this choice stems from $y^2 = R_{2n+1}(z)$ by (0.22), the identity (0.23) (which combines to $F_n H_{n+1} - G_{n-1}^2 = y^2$), and a comparison of (0.27) and (0.28). Given (0.27)–(0.31), one obtains for the divisor $(\phi(\,\cdot\,, x))$ of the meromorphic function ϕ

$$(\phi(\,\cdot\,, x)) = \mathcal{D}_{\hat{\nu}_0(x)\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_\infty \underline{\hat{\mu}}(x)}. \tag{0.32}$$

Here we abbreviated $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \mathrm{Sym}^n(\mathcal{K}_n)$, with $\mathrm{Sym}^n(\mathcal{K}_n)$ the nth symmetric product of $\mathcal{K}_n$, and used our conventions[1] (A.43), (A.47), and (A.48) to denote positive divisors of degree n and $n+1$ on $\mathcal{K}_n$. Given $\phi(\,\cdot\,, x)$, one defines the *stationary Baker–Akhiezer vector* $\Psi(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_\infty\}$ by

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1(P, x, x_0) = \exp\left(\int_{x_0}^{x} dx'\, \phi(P, x')\right), \quad \psi_2 = \psi_{1,x}.$$

In particular, this implies

$$\phi = \psi_2/\psi_1$$

and the following normalization[2] of ψ_1, $\psi_1(P, x_0, x_0) = 1$, $P \in \mathcal{K}_n \setminus \{P_\infty\}$. The Riccati-type equation (0.29) satisfied by ϕ then shows that the Baker–Akhiezer

[1] $\mathcal{D}_{\underline{Q}}(P) = m$ if P occurs m times in $\{Q_1, \dots, Q_n\}$ and zero otherwise, $\underline{Q} = \{Q_1, \dots, Q_n\} \in \mathrm{Sym}^n(\mathcal{K}_n)$. Similarly, $\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}$, $\mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_n}$, $Q_0 \in \mathcal{K}_n$, and $\mathcal{D}_Q(P) = 1$ for $P = Q$ and zero otherwise.

[2] This normalization is less innocent than it might appear at first sight. It implies that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ are the divisors of zeros and poles of $\psi_1(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_\infty\}$.

function ψ_1 is the common formal eigenfunction of the commuting pair of Lax differential expressions L and P_{2n+1},

$$L\psi_1(P) = z\psi_1(P), \tag{0.33}$$

$$P_{n+1}\psi_1(P) = iy\psi_1(P), \quad P = (z, y), \tag{0.34}$$

and at the same time the Baker–Akhiezer vector Ψ satisfies the zero-curvature equations,

$$\Psi_x(P) = U(z)\Psi(P), \tag{0.35}$$

$$iy\Psi(P) = V_{n+1}(z)\Psi(P), \quad P = (z, y). \tag{0.36}$$

Moreover, one easily verifies that away from the (finite) branch points $(E_m, 0)$, $m = 0, \dots, 2n$, of the two-sheeted Riemann surface $\mathcal{K}_n$, the two branches of ψ_1 constitute a fundamental system of solutions of (0.33) and similarly, the two branches of Ψ yield a fundamental system of solutions of (0.35). Since $\psi_1(\,\cdot\,, x, x_0)$ vanishes at $\hat{\mu}_j(x)$, $j = 1, \dots, n$ and $\psi_2(\,\cdot\,, x, x_0) = \psi_{1,x}(\,\cdot\,, x, x_0)$ vanishes at $\hat{\nu}_\ell(x)$, $\ell = 0, \dots, n$, we may call $\{\hat{\mu}_j(x)\}_{j=1,\dots,n}$ and $\{\hat{\nu}_\ell(x)\}_{\ell=0,\dots,n}$ the *Dirichlet* and *Neumann data* of L at the point $x \in \mathbb{R}$, respectively.

Now the stationary formalism is almost complete; we only need to relate the solution u of the nth stationary KdV equation and $\mathcal{K}_n$-associated data. This can be accomplished in several ways. We describe two of them next.

First we relate u and the zeros μ_j of F_n. This is easily done by comparing the coefficients of the power z^{2n} in (0.24) and results in the *trace formula*,

$$u = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j. \tag{0.37}$$

Next we will indicate how to reconstruct (at least locally) u from Dirichlet data at just one fixed point x_0. Combining the definition (0.30) of $\hat{\mu}_j$ and that of G_{n-1} in (0.10) yields, after a comparison with the x-derivative of $F_n(z, x) = \prod_{k=1}^{n}(z - \mu_k(x))$,

$$\begin{aligned} y(\hat{\mu}_j(x)) &= iG_{n-1}(\mu_j(x), x) = -(i/2)F_{n,x}(\mu_j(x), x) \\ &= (i/2)\mu_{j,x}(x)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j(x) - \mu_k(x)), \quad j = 1, \dots, n. \end{aligned}$$

Hence, one arrives at the *Dubrovin equations* for $\hat{\mu}_j$, an autonomous first-order system of differential equations on $\mathcal{K}_n$,

$$\mu_{j,x} = -2iy(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}\left(\mu_j - \mu_k\right)^{-1}, \quad j = 1, \dots, n. \tag{0.38}$$

Augmenting (0.38) with appropriate initial data

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{0.39}$$

for some $x_0 \in \mathbb{R}$, with $\mu_j(x_0)$, $j = 1, \dots, n$ assumed to be distinct, one can solve the Dubrovin system (0.38) at least locally[1] in a neighborhood of the point x_0 and then reconstruct u in that neighborhood using the trace formula (0.37). In other words, the Dirichlet data $\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n}$ in (0.39) at the point x_0 can be used to reconstruct u in a neighborhood of x_0. Since u can be shown to be meromorphic, this uniquely determines u (even though it is not necessarily clear from our discussion thus far how to reconstruct u globally). Furthermore, u satisfies $\text{s-KdV}_n(u) = 0$.

An alternative reconstruction of u, nicely complementing the one just discussed, can be given with the help of the *Riemann theta function*[2] associated with $\mathcal{K}_n$ and an appropriate homology basis of cycles on it. The known zeros and poles of ϕ (cf. (0.32)), and similarly, the set of zeros $\{\hat{\mu}_j(x)\}_{j=1,\dots,n}$ and poles $\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n}$ of the Baker–Akhiezer function $\psi_1(\,\cdot\,, x, x_0)$ together with the characteristic essential singularity of ψ_1 at P_∞, then permit one to find theta function representations for ϕ and ψ_1 by alluding to Riemann's vanishing theorem and the Riemann–Roch theorem.[3] The corresponding theta function representation of the algebro-geometric solution u of the nth stationary KdV equation then can be obtained from that of ψ_1 by an asymptotic expansion with respect to the spectral parameter near the point P_∞. Alternatively, one can use the trace formula (0.37) and apply the known theta function representations for symmetric functions of the projections $\mu_j(x)$ of the zeros $\hat{\mu}_j(x)$ of ψ_1 to the special case $\sum_{j=1}^n \mu_j(x)$ at hand. Either way, the resulting final expression for u, called the *Its–Matveev formula*, is of the type

$$u(x) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x)). \tag{0.40}$$

Here the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B} \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ (and its homology basis), and the constant $\underline{A} \in \mathbb{C}^n$ (related to the Abel map of the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$) is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \text{Sym}^n(\mathcal{K}_n)$ at the point x_0 as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ is assumed to be nonspecial.[4] Moreover, the theta function representation (0.40) remains valid as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ stays nonspecial. We emphasize the remarkable fact that the argument of the theta function in (0.40) is linear with respect to x.

[1] In some situations, such as the case of periodic u, it is possible to elevate this procedure to a global reconstruction of u even in the presence of collisions of $\hat{\mu}_j$ on $\mathcal{K}_n$. But this requires an extensive analysis we mention in the notes to Appendix F.

[2] For details on the n-dimensional theta function $\theta(\underline{z})$, $\underline{z} \in \mathbb{C}^n$, we refer to Appendices A and B.

[3] We defer the analogous discussion of ψ_2 to Chapter 1 for simplicity.

[4] If $\mathcal{D} = n_1\mathcal{D}_{Q_1} + \cdots + n_k\mathcal{D}_{Q_k} \in \text{Sym}^n(\mathcal{K}_n)$ for some $n_\ell \in \mathbb{N}$, $\ell = 1, \dots, k$, with $n_1 + \cdots + n_k = n$, then $\mathcal{D}$ is called nonspecial if there is no nonconstant meromorphic function on $\mathcal{K}_n$ that is holomorphic on $\mathcal{K}_n \setminus \{Q_1, \dots, Q_k\}$ with poles at most of order n_ℓ at Q_ℓ, $\ell = 1, \dots, k$.

The current discussion assumed that one started with a solution u of the nth stationary KdV equation and then either reconstructed it from the trace formula (0.37), or represented the given u in terms of the theta function associated with $\mathcal{K}_n$, as in (0.40). In addition to this procedure we also solve the following inverse problem: Given appropriate initial data (0.39) and solutions $\hat{\mu}_1(x), \dots, \hat{\mu}_n(x)$ of the first-order Dubrovin system (0.38) on an open interval $\Omega \subseteq \mathbb{R}$ containing the point x_0, we will define u on Ω in terms of the trace formula (0.37) and then prove that u so defined satisfies the nth stationary KdV equation on Ω.

This completes our somewhat lengthy excursion into the stationary KdV hierarchy. In the following we return to the time-dependent KdV hierarchy and describe the analogous steps involved to construct solutions $u = u(x, t_r)$ of the rth KdV equation with initial values being algebro-geometric solutions of the nth stationary KdV equation. More precisely, we are seeking a solution u of the following algebro-geometric initial value problem

$$\widetilde{\mathrm{KdV}}_r(u) = u_{t_r} - 2\tilde{f}_{r+1,x}(u) = 0, \quad u|_{t_r=t_{0,r}} = u^{(0)}, \tag{0.41}$$

$$\text{s-KdV}_n\left(u^{(0)}\right) = -2f_{n+1,x}\left(u^{(0)}\right) = 0 \tag{0.42}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$ and a fixed curve $\mathcal{K}_n$ associated with the stationary solution $u^{(0)}$ in (0.42).

We pause for a moment to reflect on the pair of equations (0.41), (0.42): As it turns out, they represent a dynamical system on the set of algebro-geometric solutions isospectral to the initial value $u^{(0)}$. The term *isospectral* here alludes to the fact that for any fixed t_r, the solution $u(\,\cdot\,, t_r)$ of (0.41), (0.42) is a stationary solution of (0.42),

$$\text{s-KdV}_n(u(\,\cdot\,, t_r)) = -2f_{n+1,x}(u(\,\cdot\,, t_r)) = 0$$

associated with the fixed underlying algebraic curve $\mathcal{K}_n$. Put differently, $u(\,\cdot\,, t_r)$ is an isospectral deformation of $u^{(0)}$ with t_r the corresponding deformation parameter. In particular, $u(\,\cdot\,, t_r)$ traces out a curve in the set of algebro-geometric solutions isospectral to $u^{(0)}$.

Since the integration constants in the functionals f_ℓ of u in the stationary and time-dependent contexts are independent of each other, we indicate this by placing a tilde over all the time-dependent quantities. Hence, we will employ the notation $\widetilde{P}_{2r+1}$, $\widetilde{V}_{r+1}$, $\widetilde{F}_r$, etc., to distinguish them from P_{2n+1}, V_{n+1}, F_n, etc. Thus, $\widetilde{P}_{2r+1}$, $\widetilde{V}_{r+1}$, $\widetilde{F}_r$, $\widetilde{H}_{r+1}$, $\tilde{f}_s$ are constructed in the same way as P_{2n+1}, V_{n+1}, F_n, H_n, f_ℓ using the recursion (0.3) with the only difference being that the set of integration constants $\tilde{c}_r$ in $\tilde{f}_s$ is independent of the set c_k used in computing f_ℓ.

Our strategy will be the same as in the stationary case: Assuming existence of a solution u, we will deduce many of its properties, which, in the end, will yield an explicit expression for the solution. In fact, we will go a step further, postulating the equations

$$u_{t_r} = -(1/2)\widetilde{F}_{r,xxx} + 2(u-z)\widetilde{F}_{r,x} + u_x\widetilde{F}_r, \tag{0.43}$$

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2 = R_{2n+1}, \tag{0.44}$$

where $u^{(0)} = u^{(0)}(x)$ in (0.42) has been replaced by $u = u(x, t_r)$ in (0.44). Here,

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell}z^\ell = \prod_{j=1}^{n}(z-\mu_j), \quad \widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s}z^s$$

for fixed $n, r \in \mathbb{N}_0$. Introducing G_{n-1}, H_{n+1}, U, V_{n+1} and $\widetilde{G}_{r-1}$, $\widetilde{H}_{r+1}$, $\widetilde{V}_{r+1}$ (replacing F_n by $\widetilde{F}_r$) as in (0.10)–(0.13), we observe that the basic equations (0.43), (0.44) are equivalent to the *Lax equations*

$$\frac{d}{dt_r}L - [\widetilde{P}_{2r+1}, L] = 0,$$

$$[P_{2n+1}, L] = 0,$$

and to the *zero-curvature equations*

$$U_{t_r} - \widetilde{V}_{r+1,x} + [U, \widetilde{V}_{r+1}] = 0, \tag{0.45}$$

$$-V_{n+1,x} + [U, V_{n+1}] = 0. \tag{0.46}$$

Moreover, one computes in analogy to (0.21) and (0.22) that

$$\partial_x \det(yI_2 - iV_{n+1}(z, x, t_r)) = 0,$$

$$\partial_{t_r} \det(yI_2 - iV_{n+1}(z, x, t_r)) = 0,$$

and hence

$$\begin{aligned}\det(yI_2 - iV_{n+1}(z, x, t_r)) &= y^2 - \det(V_{n+1}(z, x, t_r)) \\ &= y^2 + G_{n-1}(z, x, t_r)^2 - F_n(z, x, t_r)H_{n+1}(z, x, t_r) = y^2 - R_{2n+1}(z)\end{aligned} \tag{0.47}$$

is independent of (x, t_r). Thus,

$$F_nH_{n+1} - G_{n-1}^2 = R_{2n+1}, \tag{0.48}$$

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2 = R_{2n+1} \tag{0.49}$$

hold as in the stationary context. The independence of (0.47) of t_r can be interpreted as follows: The rth KdV flow represents an isospectral deformation of the curve $\mathcal{K}_n$ defined in (0.26); in particular,[1] the branch points of $\mathcal{K}_n$ remain invariant under

[1] Property (0.50) is weaker than the usually stated isospectral deformation of the Lax operator $L(t_r)$. However, the latter is a more delicate functional analytic problem marred by possible singularities of u and possible non-self-adjointness of $L(t_r)$.

these flows,

$$\partial_{t_r} E_m = 0, \quad m = 0, \dots, 2n. \tag{0.50}$$

As in the stationary case, one can now introduce the basic meromorphic function ϕ on $\mathcal{K}_n$ by

$$\begin{aligned}\phi(P, x, t_r) &= \frac{iy - G_{n-1}(z, x, t_r)}{F_n(z, x, t_r)} \\ &= \frac{-H_{n+1}(z, x, t_r)}{iy + G_{n-1,x}(z, x, t_r)}, \quad P = (z, y) \in \mathcal{K}_n,\end{aligned}$$

and a comparison with (0.45) and (0.46) then shows that ϕ satisfies the Riccati-type equations

$$\phi_x + \phi^2 = u - z, \tag{0.51}$$

$$\phi_{t_r} = \partial_x\big(\widetilde{F}_r\phi + \widetilde{G}_{r-1}\big) = -\widetilde{F}_r\phi^2 - 2\widetilde{G}_{r-1}\phi - \widetilde{H}_r. \tag{0.52}$$

Next, factorizing F_n and H_{n+1} as before,

$$F_n(z, x, t_r) = \prod_{j=1}^{n}(z - \mu_j(x, t_r)), \quad H_{n+1}(z, x, t_r) = \prod_{\ell=0}^{n}(z - \nu_\ell(x, t_r)),$$

one introduces points $\hat{\mu}_j(x, t_r)$, $\hat{\nu}_\ell(x, t_r)$ on $\mathcal{K}_n$ by

$$\begin{aligned}\hat{\mu}_j &= (\mu_j, iG_{n-1,x}(\mu_j)), \quad j = 1, \dots, n, \\ \hat{\nu}_\ell &= (\nu_\ell, -iG_{n-1,x}(\nu_\ell)), \quad \ell = 0, \dots, n\end{aligned}$$

and obtains for the divisor $(\phi(\,\cdot\,, x, t_r))$ of the meromorphic function ϕ

$$(\phi(\,\cdot\,, x, t_r)) = \mathcal{D}_{\hat{\nu}_0(x,t_r)\underline{\hat{\nu}}(x,t_r)} - \mathcal{D}_{P_\infty\underline{\hat{\mu}}(x,t_r)},$$

as in the stationary context. Given $\phi(\,\cdot\,, x, t_r)$, one then defines the *time-dependent Baker–Akhiezer vector* $\Psi(\,\cdot\,, x, x_0, t_r, t_{0,r})$ on $\mathcal{K}_n \setminus \{P_\infty\}$ by

$$\Psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix},$$

$$\psi_1(P, x, x_0, t_r, t_{0,r}) = \exp\bigg(\int_{t_{0,r}}^{t_r} ds\,(\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \widetilde{G}_{r-1}(z, x_0, s)) + \int_{x_0}^{x} dx'\,\phi(P, x', t_r)\bigg),$$

$$\psi_2 = \psi_{1,x}.$$

The Riccati-type equations (0.51), (0.52) satisfied by ϕ then show that

$$-V_{n+1,t_r} + [\widetilde{V}_{r+1}, V_{n+1}] = 0 \tag{0.53}$$

in addition to (0.45), (0.46). Moreover, they yield again that the Baker–Akhiezer function ψ_1 is the common formal eigenfunction of the commuting pair of Lax differential expressions $L(t_r)$ and $P_{2n+1}(t_r)$,

$$\begin{aligned} L\psi_1(P) &= z\psi_1(P), \\ P_{n+1}\psi_1(P) &= iy\psi_1(P), \\ \psi_{t_r}(P) &= \widetilde{P}_{2r+1}\psi(P) \\ &= \widetilde{F}_r(z)\psi_x(P) + \widetilde{G}_{r-1}(z)\psi(P), \quad P = (z, y), \end{aligned}$$

and at the same time the Baker–Akhiezer vector Ψ satisfies the zero-curvature equations

$$\begin{aligned} \Psi_x(P) &= U(z)\Psi(P), \\ iy\Psi(P) &= V_{n+1}(z)\Psi(P), \\ \Psi_{t_r}(P) &= \widetilde{V}_{r+1}(z)\Psi(P), \quad P = (z, y). \end{aligned}$$

The remaining time-dependent constructions closely follow our stationary outline. First one notes again the *trace formula*

$$u(x, t_r) = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j(x, t_r) \tag{0.54}$$

as a consequence of (0.49). Next, to reconstruct u (locally) from *Dirichlet data* at just one fixed point $(x_0, t_{0,r})$, one derives the *Dubrovin equations*[1]

$$\begin{aligned} \mu_{j,x} &= -2iy(\hat{\mu}_j)\prod_{\substack{k=1 \\ k\neq j}}^{n}(\mu_j - \mu_k)^{-1}, \\ \mu_{j,t_r} &= -2i\widetilde{F}_r(\mu_j)y(\hat{\mu}_j)\prod_{\substack{k=1 \\ k\neq j}}^{n}(\mu_j - \mu_k)^{-1}, \end{aligned} \tag{0.55}$$

using (0.44), and (0.53) for F_{n,t_r}, as in the stationary case. Augmenting (0.55) with appropriate initial data

$$\{\hat{\mu}_j(x_0, t_{0,r})\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{0.56}$$

for some $(x_0, t_{0,r}) \in \mathbb{R}^2$, with $\mu_j(x_0, t_{0,r})$, $j = 1, \dots, n$ assumed to be distinct, one can again solve the Dubrovin system (0.55), at least locally in a neighborhood of the point $(x_0, t_{0,r})$, and then reconstruct u in that neighborhood using the trace formula (0.54). In other words, the Dirichlet data $\{\hat{\mu}_j(x_0, t_{0,r})\}_{j=1,\dots,n}$ in (0.56) at the point $(x_0, t_{0,r})$ reconstruct u in a neighborhood of $(x_0, t_{0,r})$.

The corresponding representations of u, ϕ, and Ψ in terms of the *Riemann theta function* associated with $\mathcal{K}_n$ is then obtained in close analogy to the stationary case. Particularly, in the case of u, one obtains the *Its–Matveev formula*

$$u(x, t_r) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r)), \tag{0.57}$$

[1] To obtain a closed system of differential equations, one has to express $\widetilde{F}_r(\mu_j)$ solely in terms of $\mu_1, \dots, \mu_n$ and $E_0, \dots, E_{2n+1}$; see (1.222) and (1.223).

where the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B}, \underline{C}_r \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ and r, and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0, t_{0,r}) = (\hat{\mu}_1(x_0, t_{0,r}), \dots, \hat{\mu}_n(x_0, t_{0,r})) \in \mathrm{Sym}^n(\mathcal{K}_n)$ at the point $(x_0, t_{0,r})$ as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}$ is assumed to be nonspecial. Moreover, the theta function representation (0.57) remains valid as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ stays nonspecial. Again one notes the remarkable fact that the argument of the theta function in (0.57) is linear with respect to both x and t_r.

Again, the current discussion assumed one started with a solution u of the KdV initial value problem (0.41), (0.42) and then either reconstructed it from the trace formula (0.54) or represented the given u in terms of the theta function associated with $\mathcal{K}_n$, as in (0.57). In addition to this procedure we also solve the following inverse problem: Given appropriate initial data (0.56) and solutions $\hat{\mu}_1(x, t_r), \dots, \hat{\mu}_n(x, t_r)$ of the first-order Dubrovin system (0.55) on a connected open set $\Omega \subseteq \mathbb{R}^2$ containing the point $(x_0, t_{0,r})$, we will define u on Ω in terms of the trace formula (0.54) and then prove that u so defined satisfies the KdV initial value problem (0.41), (0.42) on Ω.

The reader will have noticed that we used terms such as *integrability, soliton equations, isospectral deformations*, etc., without offering a precise definition for them. Arguably, an integrable system in connection with nonlinear evolution equations should possess several properties, including, for instance,

- infinitely many conservation laws
- isospectral deformations of a Lax operator
- action-angle variables, Hamiltonian formalism
- algebraic (spectral) curves
- infinitely many symmetries and transformation groups
- "explicit" solutions.

Although many of these properties apply to particular systems of interest, there is simply no generally accepted definition to date of what constitutes an integrable system.[1] That explicit but meromorphic (i.e., singular) solutions of systems such as the KdV hierarchy abound and local integrability of conserved densities as well as the functional analytic meaning of the Lax operator and its isospectral deformations in appropriate spaces are not obvious makes it plausible that no universally accepted notion of integrability can be achieved. Thus, different schools have necessarily introduced different shades of integrability (Liouville integrability, analytic integrability, algebraically complete integrability, etc.); in this monograph we found it useful to focus on the existence of underlying algebraic curves and explicit representations of solutions in terms of corresponding Riemann theta functions and limiting situations thereof.

[1] This has been eloquently discussed in Hitchin et al. (1999, p. 1ff). Most appropriate in this context seems Cherednik's statement, "All non-integrable equations are non-integrable the same way, all integrable ones are integrable in their own way," in the preface to Cherednik (1996).