

Contents

Introduction	<i>page</i> 1
1. Overview of this book	1
2. Some detail concerning the content	4
3. Acknowledgements	5
4. Leitfaden	5
Chapter 1. Preliminaries	7
1. Hermitian forms	7
2. Reflections	9
3. Groups	12
4. Modules and representations	13
5. Irreducible unitary reflection groups	15
6. Cartan matrices	17
7. The field of definition	19
Exercises	21
Chapter 2. The groups $G(m, p, n)$	23
1. Primitivity and imprimitivity	23
2. Wreath products and monomial representations	24
3. Properties of the groups $G(m, p, n)$	25
4. The imprimitive unitary reflection groups	27
5. Imprimitive subgroups of primitive reflection groups	32
6. Root systems for $G(m, p, n)$	34
7. Generators for $G(m, p, n)$	35
8. Invariant polynomials for $G(m, p, n)$	36
Exercises	37
Chapter 3. Polynomial invariants	39
1. Tensor and symmetric algebras	39
2. The algebra of invariants	41
3. Invariants of a finite group	42
4. The action of a reflection	46

vi	Contents	
	5. The Shephard–Todd–Chevalley Theorem	46
	6. The coinvariant algebra	51
	Exercises	53
Chapter 4.	Poincaré series and characterisations of reflection groups	54
	1. Poincaré series	54
	2. Exterior and symmetric algebras and Molien’s Theorem	56
	3. A characterisation of finite reflection groups	61
	4. Exponents	63
	Exercises	65
Chapter 5.	Quaternions and the finite subgroups of $SU_2(\mathbb{C})$	66
	1. The quaternions	67
	2. The groups $O_3(\mathbb{R})$ and $O_4(\mathbb{R})$	69
	3. The groups $SU_2(\mathbb{C})$ and $U_2(\mathbb{C})$	71
	4. The finite subgroups of the quaternions	72
	5. The finite subgroups of $SO_3(\mathbb{R})$ and $SU_2(\mathbb{C})$	77
	6. Quaternions, reflections and root systems	79
	Exercises	83
Chapter 6.	Finite unitary reflection groups of rank two	84
	1. The primitive reflection subgroups of $U_2(\mathbb{C})$	84
	2. The reflection groups of type \mathcal{T}	85
	3. The reflection groups of type \mathcal{O}	87
	4. The reflection groups of type \mathcal{I}	89
	5. Cartan matrices and the ring of definition	90
	6. Invariants	93
	Exercises	98
Chapter 7.	Line systems	99
	1. Bounds on line systems	99
	2. Star-closed Euclidean line systems	100
	3. Reflections and star-closed line systems	101
	4. Extensions of line systems	103
	5. Line systems for imprimitive reflection groups	104
	6. Line systems for primitive reflection groups	105
	7. The Goethals–Seidel decomposition for 3-systems	111
	8. Extensions of $\mathcal{D}_n^{(2)}$ and $\mathcal{D}_n^{(3)}$	115
	9. Further structure of line systems in \mathbb{C}^n	119
	10. Extensions of Euclidean line systems	120
	11. Extensions of \mathcal{A}_n , \mathcal{E}_n and \mathcal{K}_n in \mathbb{C}^n	125
	12. Extensions of 4-systems	127
	Exercises	133

Contents	vii
Chapter 8. The Shephard and Todd classification	137
1. Outline of the classification	137
2. Blichfeldt's Theorem	138
3. Consequences of Blichfeldt's Theorem	140
4. Extensions of 5-systems	142
5. Line systems and reflections of order three	146
6. Extensions of ternary 6-systems	149
7. The classification	151
8. Root systems and the ring of definition	153
9. Reduction modulo p	155
10. Identification of the primitive reflection groups	157
Exercises	168
Chapter 9. The orbit map, harmonic polynomials and semi-invariants	171
1. The orbit map	171
2. Skew invariants and the Jacobian	172
3. The rank of the Jacobian	174
4. Semi-invariants	176
5. Differential operators	179
6. The space of G -harmonic polynomials	183
7. Steinberg's fixed point theorem	186
Exercises	189
Chapter 10. Covariants and related polynomial identities	191
1. The space of covariants	191
2. Gutkin's Theorem	194
3. Differential invariants	198
4. Some special cases of covariants	199
5. Two-variable Poincaré series and specialisations	201
Exercises	206
Chapter 11. Eigenspace theory and reflection subquotients	208
1. Basic affine algebraic geometry	208
2. Eigenspaces of elements of reflection groups	212
3. Reflection subquotients of unitary reflection groups	213
4. Regular elements	215
5. Properties of the reflection subquotients	218
6. Eigenvalues of pseudoregular elements	222
Chapter 12. Reflection cosets and twisted invariant theory	228
1. Reflection cosets	228
2. Twisted invariant theory	229
3. Eigenspace theory for reflection cosets	231

viii	Contents	
	4. Subquotients and centralisers	237
	5. Parabolic subgroups and the coinvariant algebra	239
	6. Duality groups	242
	Exercises	244
	Appendix A. Some background in commutative algebra	246
	Appendix B. Forms over finite fields	250
	1. Basic definitions	250
	2. Witt's Theorem	251
	3. The Wall form, the spinor norm and Dickson's invariant	251
	4. Order formulae	252
	5. Reflections in finite orthogonal groups	253
	Appendix C. Applications and further reading	255
	1. The space of regular elements	255
	2. Fundamental groups, braid groups, presentations	258
	3. Hecke algebras	261
	4. Reductive groups over finite fields	266
	Appendix D. Tables	271
	1. The primitive unitary reflection groups	272
	2. Degrees and codegrees	274
	3. Cartan matrices	276
	4. Maximal subsystems	277
	5. Reflection cosets	277
	Bibliography	279
	Index of notation	289
	Index	291