Unitary Reflection Groups

A unitary reflection is a linear transformation of a complex vector space which fixes each point in a hyperplane. Intuitively, it resembles the transformation an image undergoes when it is viewed through a kaleidoscope, or arrangement of mirrors.

This book gives a complete classification of all finite groups which are generated by unitary reflections, using the method of line systems. Irreducible groups are studied in detail, and are identified with finite linear groups. The new invariant theoretic proof of Steinberg's fixed point theorem is treated fully. The same approach is used to develop the theory of eigenspaces of elements of reflection groups and their twisted analogues. This includes an extension of Springer's theory of regular elements to reflection cosets. An appendix outlines links to representation theory, topology and mathematical physics.

Containing over 100 exercises ranging in difficulty from elementary to research level, this book is ideal for honours and graduate students, or for researchers in algebra, topology and mathematical physics.

AUSTRALIAN MATHEMATICAL SOCIETY LECTURE SERIES

1 Introduction to Linear and Convex Programming, N. CAMERON
2 Manifolds and Mechanics, A. JONES, A. GRAY & R. HUTTON
3 Introduction to the Analysis of Metric Spaces, J. R. GILES
4 An Introduction to Mathematical Physiology and Biology, J. MAZUMDAR
5 2-Knots and their Groups, J. HILLMAN
6 The Mathematics of Projectiles in Sport, N. DE MESTRE
7 The Petersen Graph, D. A. HOLTON & J. SHEEHAN
8 Low Rank Representations and Graphs for Sporadic Groups, C. E. PRAEGER & L. H. SOICHER
9 Algebraic Groups and Lie Groups, G. I. LEHRER (ed.)
10 Modelling with Differential and Difference Equations, G. FULFORD, P. FORRESTER & A. JONES
11 Geometric Analysis and Lie Theory in Mathematics and Physics, A. L. CAREY & M. K. MURRAY (eds.)
12 Foundations of Convex Geometry, W. A. COPPEL
13 Introduction to the Analysis of Normed Linear Spaces, J. R. GILES
14 Integral: An Easy Approach after Kurzweil and Henstock, L. P. YEE & R. VYBORNY
15 Geometric Approaches to Differential Equations, P. J. VASSILIOU & I. G. LISLE (eds.)
16 Industrial Mathematics, G. R. FULFORD & P. BROADBRIDGE
17 A Course in Modern Analysis and its Applications, G. COHEN
18 Chaos: A Mathematical Introduction, J. BANKS, V. DRAGAN & A. JONES
19 Quantum Groups, R. STREET

Australian Mathematical Society Lecture Series: 20

Unitary Reflection Groups

GUSTAV I. LEHRER
DONALD E. TAYLOR
School of Mathematics and Statistics
University of Sydney

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05-06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521749893

First published 2009

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging in Publication data
Lehrer, Gus, 1947–
Unitary reflection groups / Gustav I. Lehrer, Donald E. Taylor.
p.   cm. – (Australian Mathematical Society lecture series ; 20)
Includes bibliographical references and indexes.
ISBN 978-0-521-74989-3 (hardback : alk. paper)
1. Group theory – Reflections.   I. Taylor, Donald E. (Donald Ewen), 1945–   II. Title.
QA174.2.L44   2009
512´.2 – dc22   2008056031

ISBN 978-0-521-74989-3 Paperback

Contents

Introduction *page* 1
1. Overview of this book 1
2. Some detail concerning the content 4
3. Acknowledgements 5
4. Leitfaden 5

Chapter 1. Preliminaries 7
1. Hermitian forms 7
2. Reflections 9
3. Groups 12
4. Modules and representations 13
5. Irreducible unitary reflection groups 15
6. Cartan matrices 17
7. The field of definition 19
Exercises 21

Chapter 2. The groups $G(m, p, n)$ 23
1. Primitivity and imprimitivity 23
2. Wreath products and monomial representations 24
3. Properties of the groups $G(m, p, n)$ 25
4. The imprimitive unitary reflection groups 27
5. Imprimitive subgroups of primitive reflection groups 32
6. Root systems for $G(m, p, n)$ 34
7. Generators for $G(m, p, n)$ 35
8. Invariant polynomials for $G(m, p, n)$ 36
Exercises 37

Chapter 3. Polynomial invariants 39
1. Tensor and symmetric algebras 39
2. The algebra of invariants 41
3. Invariants of a finite group 42
4. The action of a reflection 46

5. The Shephard–Todd–Chevalley Theorem 46
6. The coinvariant algebra 51
Exercises 53

Chapter 4. Poincaré series and characterisations of reflection groups 54
1. Poincaré series 54
2. Exterior and symmetric algebras and Molien's Theorem 56
3. A characterisation of finite reflection groups 61
4. Exponents 63
Exercises 65

Chapter 5. Quaternions and the finite subgroups of $SU_2(\mathbb{C})$ 66
1. The quaternions 67
2. The groups $O_3(\mathbb{R})$ and $O_4(\mathbb{R})$ 69
3. The groups $SU_2(\mathbb{C})$ and $U_2(\mathbb{C})$ 71
4. The finite subgroups of the quaternions 72
5. The finite subgroups of $SO_3(\mathbb{R})$ and $SU_2(\mathbb{C})$ 77
6. Quaternions, reflections and root systems 79
Exercises 83

Chapter 6. Finite unitary reflection groups of rank two 84
1. The primitive reflection subgroups of $U_2(\mathbb{C})$ 84
2. The reflection groups of type $\mathcal{T}$ 85
3. The reflection groups of type $\mathcal{O}$ 87
4. The reflection groups of type $\mathcal{I}$ 89
5. Cartan matrices and the ring of definition 90
6. Invariants 93
Exercises 98

Chapter 7. Line systems 99
1. Bounds on line systems 99
2. Star-closed Euclidean line systems 100
3. Reflections and star-closed line systems 101
4. Extensions of line systems 103
5. Line systems for imprimitive reflection groups 104
6. Line systems for primitive reflection groups 105
7. The Goethals–Seidel decomposition for 3-systems 111
8. Extensions of $\mathcal{D}_n^{(2)}$ and $\mathcal{D}_n^{(3)}$ 115
9. Further structure of line systems in $\mathbb{C}^n$ 119
10. Extensions of Euclidean line systems 120
11. Extensions of $\mathcal{A}_n$, $\mathcal{E}_n$ and $\mathcal{K}_n$ in $\mathbb{C}^n$ 125
12. Extensions of 4-systems 127
Exercises 133

Chapter 8. The Shephard and Todd classification 137
1. Outline of the classification 137
2. Blichfeldt's Theorem 138
3. Consequences of Blichfeldt's Theorem 140
4. Extensions of 5-systems 142
5. Line systems and reflections of order three 146
6. Extensions of ternary 6-systems 149
7. The classification 151
8. Root systems and the ring of definition 153
9. Reduction modulo p 155
10. Identification of the primitive reflection groups 157
Exercises 168

Chapter 9. The orbit map, harmonic polynomials and semi-invariants 171
1. The orbit map 171
2. Skew invariants and the Jacobian 172
3. The rank of the Jacobian 174
4. Semi-invariants 176
5. Differential operators 179
6. The space of G-harmonic polynomials 183
7. Steinberg's fixed point theorem 186
Exercises 189

Chapter 10. Covariants and related polynomial identities 191
1. The space of covariants 191
2. Gutkin's Theorem 194
3. Differential invariants 198
4. Some special cases of covariants 199
5. Two-variable Poincaré series and specialisations 201
Exercises 206

Chapter 11. Eigenspace theory and reflection subquotients 208
1. Basic affine algebraic geometry 208
2. Eigenspaces of elements of reflection groups 212
3. Reflection subquotients of unitary reflection groups 213
4. Regular elements 215
5. Properties of the reflection subquotients 218
6. Eigenvalues of pseudoregular elements 222

Chapter 12. Reflection cosets and twisted invariant theory 228
1. Reflection cosets 228
2. Twisted invariant theory 229
3. Eigenspace theory for reflection cosets 231

4. Subquotients and centralisers 237
5. Parabolic subgroups and the coinvariant algebra 239
6. Duality groups 242
Exercises 244

Appendix A. Some background in commutative algebra 246

Appendix B. Forms over finite fields 250
1. Basic definitions 250
2. Witt's Theorem 251
3. The Wall form, the spinor norm and Dickson's invariant 251
4. Order formulae 252
5. Reflections in finite orthogonal groups 253

Appendix C. Applications and further reading 255
1. The space of regular elements 255
2. Fundamental groups, braid groups, presentations 258
3. Hecke algebras 261
4. Reductive groups over finite fields 266

Appendix D. Tables 271
1. The primitive unitary reflection groups 272
2. Degrees and codegrees 274
3. Cartan matrices 276
4. Maximal subsystems 277
5. Reflection cosets 277

Bibliography 279

Index of notation 289

Index 291