Introduction

1. Overview of this book

In real Euclidean space a *reflection* is an orthogonal transformation which fixes every vector of some hyperplane, i.e. a subspace of codimension one. Thus a real reflection necessarily has order two. Finite groups generated by reflections in a real vector space have been studied in great depth and they play a central rôle in many branches of mathematics, particularly in the theory of Lie groups and Lie algebras, where many of them appear as 'Weyl groups'. They might be thought of as linking the discrete and continuous strands of Felix Klein's Erlangen programme, according to which geometry is studied through the group of symmetries of the space concerned. The standard work on these groups is Bourbaki's treatise [**33**] of 1968 and there is a more recent account in the monograph of Humphreys [**119**]. See [**110**] for a survey of the breadth of applications up to 1977.

In 1951 Shephard (see [**191**] and [**192**]) extended the concept of reflection to a complex vector space with an hermitian inner product. A reflection (sometimes called a pseudo-reflection) is a linear transformation of finite order, which fixes a hyperplane pointwise. Almost immediately, Shephard and Todd [**193**], building on the work of many authors over the preceding century, obtained the complete classification of finite groups generated by (unitary) reflections. These groups include the Euclidean reflection groups, and arise naturally from them when one considers certain subgroups and subquotients which act on subspaces of the complexification of the real space with which one begins. These more general 'unitary reflection groups' have a wide range of applications, including

(*i*) the structure and representation theory of reductive algebraic groups;
(*ii*) Hecke algebras;
(*iii*) knot theory;
(*iv*) moduli spaces;
(*v*) algebraic topology, particularly in low dimensions;
(*vi*) invariant theory and algebraic geometry;
(*vii*) differential equations;
(*viii*) mathematical physics.

In writing this book we have had four principal objectives in mind. Firstly, although it is now more than half a century since the Shephard–Todd classification, there is still no complete and coherent account of this classification in book form in the literature, although there have been some research articles, e.g. [**54, 55**], which have addressed the subject. This is in sharp contrast to the situation for the real reflection groups, which are precisely the finite Coxeter groups (see [**209**, Appendix, Theorem 38] and [**33**, Chap. V, Th. 1 et 2]), and whose classification generally proceeds through the classification of root systems, which is readily available in the literature. Taking into account that the original Shephard–Todd classification itself depends on a significant body of earlier literature, and that the classification is much used and referred to, we thought it useful to provide a complete treatment of the classification of the unitary reflection groups.

For any unitary reflection group G, there is a corresponding collection of lines in the ambient complex space V, obtained by taking the lines orthogonal to the reflecting hyperplanes of G. We call this collection a '*line system*'; our treatment of the classification of the unitary reflection groups comes down to a classification of line systems. There are interrelationships among the various irreducible reflection groups, which may be studied through the relationships among their line systems. A consequence of our approach to the classification is that we are able to elucidate these systematically. In particular we indicate all the maximal reflection subgroups of any irreducible group, which of course essentially provides a complete list of reflection subgroups of each irreducible group. For analogous information concerning the real groups the classical references are [**31, 92, 90**]. More generally, we have sought to provide a good deal of detail concerning individual groups. We have also provided identifications of the irreducible groups with linear groups over finite fields where appropriate.

Our second objective relates to the invariant theory of unitary reflection groups. It is a beautiful result of Shephard and Todd that the unitary reflection groups are characterised among all complex linear groups as those whose algebra of invariants is free, or equivalently those which have a smooth variety of orbits on the vector space V in which they act. This is the merest hint that the invariant theory of unitary reflection groups is a rich vein for study. In this book we develop this theory in several directions. We give a complete treatment of the M-exponents of G, for any G-module M; this includes the usual exponents and the more recent 'coexponents', which are closely related to the topology of the complement M_G of the reflecting hyperplanes of G. These ideas are used to study parabolic subgroups, i.e. the stabilisers of points (or subspaces) of V; in particular, we give a simple proof of Steinberg's Theorem that the parabolic subgroups are reflection groups.

We give a comprehensive account of the application of invariant theoretic methods to the eigenspace theory of Springer and Lehrer–Springer. This has obviated the need for intersection theory, and requires only elementary concepts from affine algebraic geometry, which we provide in Appendix A. Our account includes material concerning centralisers of elements of reflection groups, which we regard as an integral part of the theory. In a related circle of ideas, we study harmonic polynomial functions on V through duality between the polynomial functions and differential operators. These two themes are united in applications to the structure of the coinvariant algebra. In particular, we prove results relating its module structure to that of parabolic subgroups of G.

Thirdly, in the study of reflection groups, it becomes apparent early, even if one confines attention to real groups, that it is important to consider situations where there is a linear transformation γ of the ambient space V which normalises the reflection group G. Examples include normalisers of parabolic subgroups, the ramification groups occurring in the representation theory of reductive groups, and many of the applications in the areas outlined in Appendix C. In view of this, we define 'reflection cosets' γG, and provide a chapter (12) on the 'twisted invariant theory' of such cosets. This theory is very close to the untwisted case, with only a certain set of roots of unity, the 'M-factors' for each $\langle \gamma, G \rangle$-module M entering the picture. The study of these fits well with the eigenspace theory alluded to above.

Finally, although the purpose of this book is to provide background in the core material on reflection groups, we are very conscious that current interest in this subject arises from its application to many and varied branches of mathematics, including those listed above. We have therefore provided, in Appendix C, a brief outline of how the subject matter in this book applies to various areas. We have attempted to write our development in such a way as to be accessible to people working in the diverse areas in which it may be applied. This appendix also contains a number of questions and open problems, which are suitable as research topics.

The reader is referred to the appendix for details, and we confine ourselves here to the following remark as to how these applications arise. A key observation which leads to links between the theory of reflection groups and other areas is that there is an important topological space associated with any unitary reflection group G, namely its associated hyperplane complement M_G, which is defined as the set of points of V which lie on no reflecting hyperplane of G. In the case where G is the symmetric group $\mathrm{Sym}(n)$, M_G is the space of ordered configurations $(z_1, z_2, \ldots, z_n)$ of distinct points $z \in \mathbb{C}$. Now M_G, and its quotient X_G by G, have the structure of complex analytic manifolds, but may also be regarded as the varieties of complex points of algebraic schemes over a number field. Moreover X_G has an interesting fundamental group, which in the example of $\mathrm{Sym}(n)$ is the classical Artin braid group. One may therefore consider differential equations for functions on X_G, or the geometry of its points over various rings; moreover the group algebra of its fundamental group has quotients which arise in various ways in the representation theory of reductive groups. It is these various ways of regarding M_G, X_G and associated spaces which lead to applications in many and varied areas of mathematics.

2. Some detail concerning the content

In this section we provide a brief description of the material in this book, chapter by chapter. In the next section we indicate the logical interdependencies among the chapters, and make some suggestions as to how the book may be used as a text for courses.

In Chapter 1, we introduce the elementary notions which underlie the subject, and define, for any unitary reflection group, the basic concepts of root, root system, and Cartan matrix. These are used later in the classification. In Chapter 2, we make a fairly detailed study of the imprimitive groups $G(m, p, n)$. The Shephard–Todd classification shows that any irreducible unitary reflection group is either one of these groups, or one of the 34 'exceptional groups', which were denoted in [**193**] as $G_4, G_5, \ldots, G_{37}$, a notation which is still commonly used today. Of these, 19 are two-dimensional, and Chapters 5 and 6 are devoted to their description and classification.

Chapters 3 and 4 provide the characterisation of reflection groups as precisely those groups with a free algebra of invariants. The former gives a general introduction to invariant theory and multilinear algebra, and introduces the coinvariant algebra for the first time; this is used to define the χ-exponents of G and the fake degree of any character χ of G. Chapter 4 uses Poincaré series to complete the proof of the Shephard–Todd characterisation.

In Chapters 7 and 8 the classification of the irreducible unitary reflection groups is completed. First, in Chapter 7, line systems are defined and studied in detail. It is explained how they may be extended, and what restrictions there are on them. In Chapter 8, a complete classification is given of all permissable line systems, and interrelationships among them. This is used to complete the classification. An interesting by-product of our development is the fact that any reflection group may be written over the ring of integers in the field generated by the character values of its defining representation. We call this the *ring of definition* of G, and show that it plays an important role in the description of reflection subgroups, and line subsystems.

The next two chapters, 9 and 10, provide a deeper study of the relationship between the structure and representations of G and its invariant theory. The orbit map $V \to V/G$ is studied in Chapter 9, and used to prove Steinberg's fixed point theorem and to study the semi-invariants of G. The space of G-harmonic functions is introduced here via duality and differential operators. The structure of the spaces of G-covariants for various representations M of G is studied in Chapter 10, and the M-exponents are defined. The usual exponents and coexponents are treated as special cases, and the structure theorems are translated here into statements concerning two-variable Poincaré series. In Chapter 11, all this is applied to give a complete treatment of the eigenspace theory of Springer and Lehrer–Springer, including related material on centralisers of elements of G.

Chapter 12 presents the twisted theory for reflection cosets which was mentioned above.

This book is intended to be suitable for a graduate student with a good background in undergraduate algebra. The books of Lang [**142**] and Atiyah–Macdonald [**8**] are more than adequate for our purpose, but we do not assume their content. On the few occasions where a little more background is required, we generally refer to these sources. The first two appendices, A and B contain background material necessary for some of the proofs in the text. The first contains some elementary affine algebraic geometry, which is needed in the exposition of the material on eigenspace theory. The second contains some material on the spinor norm which is used in the identification of some reflection groups as linear groups over finite fields.

Appendix C provides an introduction to some of the applications of the theory expounded in this book to various areas of algebra, topology and mathematical physics, and contains some suggestions for further reading. It also contains suggested research projects. Finally, Appendix D contains tables of various properties and invariants associated with irreducible finite unitary reflection groups.

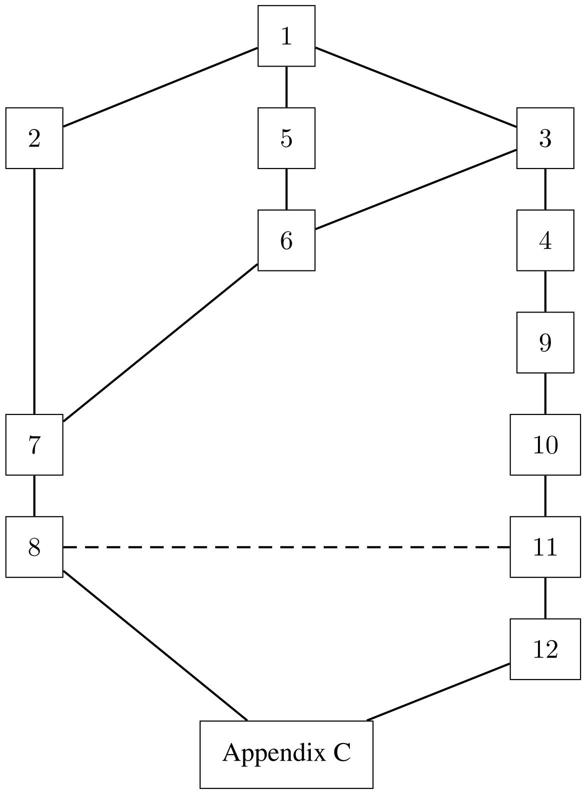
3. Acknowledgements

This book is an outgrowth of several courses given over the years to honours and postgraduate students at the University of Sydney by the first author. The first of these was given in the spring of 1995, and we thank the students of those classes for their contribution to the shaping of the ideas presented here.

The authors also thank Michel Broué and Jean Michel for extensive discussions about the unitary reflection groups, particularly their applications to representation theory, Hecke algebras and related geometric themes, and Jean Michel for his valuable comments on the manuscript.

4. Leitfaden

The logical interdependencies among the chapters are indicated in the diagram below. From this diagram, it is clear that there are two main lines of development, one for the classification and specific properties of the various groups, and the other for the invariant theoretic ideas and their application to eigenspace theory. Either of these would be suitable for a one-semester course for graduate students; alternatively one could treat a subset of the chapters at the top of the diagram below, ensuring only that if a course includes a chapter, it should also include those above it in the diagram.



CHAPTER 1

Preliminaries

In this chapter we define unitary reflections and prove some elementary facts about them. We then introduce the important concepts of root, Cartan matrix and root system, which are used extensively in our development of the classification. The notion of the Weyl group of a Cartan matrix is discussed in the context of unitary reflection groups, and some elementary properties of root systems are pointed out. We also review the basic facts and terminology of group theory and representation theory needed throughout the book.

1. Hermitian forms

Definition 1.1. Given a vector space V of dimension n over the complex field $\mathbb{C}$, an *hermitian form* on V is a mapping

$$(-,-) : V \times V \to \mathbb{C}$$

such that

$$\begin{aligned} (v_1 + v_2, w) &= (v_1, w) + (v_2, w) \\ (av, w) &= a(v, w) \\ \overline{(v, w)} &= (w, v) \end{aligned}$$

for all v, w, v_1, $v_2 \in V$ and $a \in \mathbb{C}$. The hermitian form is *positive definite* if

$$\begin{aligned} &(v, v) \geq 0 \quad \text{and} \\ &(v, v) = 0 \quad \text{if and only if} \quad v = 0. \end{aligned}$$

A positive definite hermitian form is also known as an *inner product*. For example, if V has a basis $e_1, e_2, \ldots, e_n$, we may define a positive definite hermitian form on V by

$$(u, v) := a_1\overline{b}_1 + a_2\overline{b}_2 + \cdots + a_n\overline{b}_n, \tag{1.2}$$

where $u := a_1e_1 + a_2e_2 + \cdots + a_ne_n$ and $v := b_1e_1 + b_2e_2 + \cdots + b_ne_n$. It is an easy exercise to show that every positive definite hermitian form on V can be described in this fashion with respect to a suitable basis. In other words, if $(-,-)$ and $[-,-]$ are two positive definite hermitian forms on V, then they are *equivalent* in the sense that there is an invertible linear transformation $\varphi : V \to V$ such that $(u, v) = [\varphi(u), \varphi(v)]$ for all $u, v \in V$.

Let $GL(V)$ be the group of all invertible linear transformations of V. A subgroup G of $GL(V)$ is said to leave the form $(-, -)$ *invariant* if

$$(gv, gw) = (v, w) \quad \text{for all } g \in G \text{ and for all } v, w \in V.$$

We also say that $(-, -)$ is a G-*invariant* form.

Lemma 1.3. *If G is a finite subgroup of $GL(V)$, there exists a G-invariant positive definite hermitian form on V.*

Proof. Choose a positive definite hermitian form $[-, -]$ on V and define a new form by

$$(v, w) := \sum_{g \in G} [gv, gw].$$

Then $(-, -)$ is easily seen to be hermitian and we have

$$(v, v) = \sum_{g \in G} [gv, gv] \geq 0.$$

This expression is 0 if and only if all $[gv, gv]$ are 0. Thus $(-, -)$ is positive definite. Finally, if $h \in G$, then as g runs through G, so does gh and therefore

$$(hv, hw) = \sum_{g \in G} [ghv, ghw] = \sum_{g \in G} [gv, gw] = (v, w).$$

Thus $(-, -)$ is G-invariant. □

If $(-, -)$ is any positive definite hermitian form on V, we say that $x \in GL(V)$ is *unitary* (or an *isometry*) if $(xv, xw) = (v, w)$ for all $v, w \in V$; that is, $(-, -)$ is $\langle x \rangle$-invariant, where $\langle x \rangle$ is the cyclic group generated by x.

A basis $e_1, e_2, \dots, e_n$ for V is *orthogonal* if $(e_i, e_j) = 0$ for all $i \neq j$; it is *orthonormal* if in addition $(e_i, e_i) = 1$ for all i.

Let M be the matrix of $x \in GL(V)$ with respect to an orthonormal basis of V. Then x is unitary if and only if M is a *unitary matrix*; i.e. $M\overline{M}^t = I$, where $\overline{M}^t$ denotes the transpose of the complex conjugate of M and I is the identity matrix.

The group of all isometries of V is denoted by $U(V)$ and called the *unitary group* of the form. Its subgroup of transformations of determinant 1 is called the *special unitary group*. The corresponding groups of unitary matrices will be denoted by $U_n(\mathbb{C})$ and $SU_n(\mathbb{C})$, where $n := \dim V$. The group $U(V)$ depends on the form but as any two positive definite hermitian forms on V are equivalent, $U(V)$ is unique up to conjugacy in $GL(V)$. With this notation Lemma 1.3 says that any finite subgroup of $GL(V)$ is a subgroup of $U(V)$ for an appropriate hermitian form.

2. Reflections

Throughout this section V denotes a vector space of dimension n with a positive definite hermitian form $(-,-)$.

Definition 1.4. If U is a subset of V we define the *orthogonal complement* of U to be the subspace $U^\perp := \{ v \in V \mid (u,v) = 0 \text{ for all } u \in U \}$.

If U and W are subspaces of V, we write $V = U \perp W$ to indicate that $V = U \oplus W$ and $(u,w) = 0$ for all $u \in U$ and $w \in W$. It is an easy exercise to check that $V = U \perp W$ if and only if $W = U^\perp$. Further, $U^{\perp\perp} = U$ and $\dim U + \dim U^\perp = \dim V$ for any subspace $U \subseteq V$.

Definition 1.5. Let 1 be the identity element of $GL(V)$. For $g \in GL(V)$ and $H \subseteq GL(V)$, put

(*i*) $\operatorname{Fix} g := \operatorname{Ker}(1-g) = \{ v \in V \mid gv = v \}$,
(*ii*) $V^H := \operatorname{Fix}_V(H) := \{ v \in V \mid hv = v \text{ for all } h \in H \}$, and
(*iii*) $[V,g] := \operatorname{Im}(1-g)$.

Lemma 1.6. *If* $g \in U(V)$*, then* $[V,g] = (\operatorname{Fix} g)^\perp$.

Proof. Suppose that $u := (1-g)w$ and that $v \in \operatorname{Fix} g$. Then

$$\begin{aligned}(u,v) &= (w - gw, v) = (w,v) - (gw,v)\\ &= (gw,gv) - (gw,v) = (gw, gv - v) = 0.\end{aligned}$$

Thus $[V,g] \subseteq (\operatorname{Fix} g)^\perp$ and on comparing dimensions we see that equality holds. □

Definition 1.7. A linear transformation g is a *reflection* if the order of g is finite and if $\dim[V,g] = 1$. (In some references, such as Bourbaki [**33**], such a transformation is called a pseudo-reflection.)

If g is a reflection, the subspace $\operatorname{Fix} g$ is a hyperplane, called the *reflecting hyperplane* of g.

If a spans $[V,g]$, then for all $v \in V$, there exists $\varphi(v) \in \mathbb{C}$ such that $v - gv = \varphi(v)a$. It is clear that $\varphi : V \to C$ is a linear functional such that $\operatorname{Fix} g = \operatorname{Ker} \varphi$.

We call g a *unitary* reflection if it preserves the hermitian form $(-,-)$. In this case $\operatorname{Fix} g$ is orthogonal to $[V,g]$ and $V = [V,g] \perp \operatorname{Fix} g$.

Suppose $g \in GL(V)$ is a reflection of order m. Then the cyclic group $\langle g \rangle$ has order m and hence, by Lemma 1.3, it leaves invariant a positive definite hermitian form. Thus every reflection g is a unitary reflection with respect to some form. If $H = \operatorname{Fix} g$, then g leaves invariant the line (one-dimensional subspace) $H^\perp$. Hence with respect to a basis adapted to the decomposition $V = H^\perp \perp H$, g has matrix $\operatorname{diag}[\zeta, 1, \ldots, 1]$, where ζ is a primitive m^{th} root of unity.

Definition 1.8. A *root* of a line ℓ of V is any non-zero vector of ℓ. If g is a unitary reflection, a *root* of g is a root of the line $[V, g]$. A root a is *short*, *long* or *tall* if (a, a) is 1, 2 or 3, respectively. For the most part we consider only short roots. However, in Chapters 2, 7 and 8 it will be useful to use roots of other lengths.

Any line in $\mathbb{C}^n$ contains long, short and tall roots, each of which is unique up to multiplication by an element of $S^1 := \{ z \in \mathbb{C} \mid |z| = 1 \}$.

Lemma 1.9. *If $g, h \in GL(V)$, then $\mathrm{Fix}(ghg^{-1}) = g \,\mathrm{Fix}\, h$. In particular, if r is a reflection with reflecting hyperplane $H := \mathrm{Fix}\, r$, then grg^{-1} is a reflection with reflecting hyperplane $gH = \mathrm{Fix}(grg^{-1})$.*

Definition 1.10. A *unitary reflection group* is a finite subgroup of $U(V)$ that is generated by reflections. These groups are also referred to by several authors as *complex* reflection groups.

Because of Lemma 1.3, every finite subgroup of $GL(V)$ that is generated by reflections is a unitary reflection group with respect to some positive definite hermitian form on V.

It is important to note that the concept 'unitary reflection group' includes the representation as well as the group. A given group may act as a reflection group or otherwise. For example, for $\zeta := \exp(2\pi i/m)$, the element $\mathrm{diag}[\zeta, 1, \ldots, 1]$ generates a cyclic reflection group of order m, but the (isomorphic) group generated by $\mathrm{diag}[\zeta, \zeta, 1, \ldots, 1]$ is not a reflection group.

Remark 1.11. The sentence 'G is a unitary reflection group in V' will indicate that G is a finite group, generated by reflections in V. By Lemma 1.3, there is then a positive definite G-invariant hermitian form on V, and by Corollary 1.26, this form is unique up to a non-zero positive multiple if G acts irreducibly on V.

Example 1.12. If ω is a cube root of unity, the matrices

$$r := \begin{bmatrix} \omega & 0 \\ -\omega^2 & 1 \end{bmatrix} \quad \text{and} \quad s := \begin{bmatrix} 1 & \omega^2 \\ 0 & \omega \end{bmatrix}$$

are reflections of order 3 and they generate a group of order 24. This is the group G_4 in the list of Shephard and Todd [**193**]. See the exercises at the end of the chapter for further details.

Definition 1.13. The *dual space* of V is the vector space V^* of all linear maps $\varphi : V \to \mathbb{C}$ with addition and multiplication by scalars given by

$$(\varphi + \psi)(v) := \varphi(v) + \psi(v)$$
$$(\alpha\varphi)(v) := \alpha\varphi(v).$$

The elements of V^* are sometimes referred to as (linear) functionals.