ONE

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Cantor's Paradise

Cantor's quarry was the infinite. The mathematics of number had always been about objects of which there are infinitely many, like natural numbers, or objects of which not only are there infinitely many but each is also itself infinite, like real numbers with endless decimal expansions. The infinities of geometry, like the infinity of points on a line or triangles in a plane, had always been there, but the applications of the calculus in geometry made its infinities more salient. The recognition of the infinity of its subject matter was always a reason not to test the conjectures of mathematics by checking the examples but rather to prefer proof. Aristotle urged that the infinite could only ever be potential, like a process with no fixed end, but that completed actual infinite wholes were ruled out. Such views look to countenance possibilities that could not be actual, which sounds contradictory, but even Gauss, the prince of mathematicians, had a *horror infiniti*. Cantor swam against the tide.

To work out a theory of the infinite per se, Cantor needed to figure out which things are classified as finite or infinite. That is one source of his interest in sets. For this purpose sets should be any old collections, whether unified by having something in common or not, like the Walrus's shoes and ships and cabbages and kings. Sets should be an utterly general sort, so whether there are infinitely many such and suches can always be re-asked as whether the set of such and suches is infinite. As horses are the kind that divides into stallions and mares, so sets are the kind that divides into finite and infinite.

Cantor distinguished between two sorts of infinity, one where order is front and center, and another where it is less obvious. (Order can often be taken as process finished and complete.) Since order is an extra, let us first put it aside. We want to articulate what it is for two sets to be the same in size. There are as many digits on your left hand as on your

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More information

right. One way to check this is to count each and get the same answer, five, in both cases, but that procedure assumes number, something we also want to articulate. Another way is to match your digits one-to-one, so that each is matched with just one partner. This you can do without counting or numerical claims. Order does not matter. Palm to palm, you can match thumb to thumb, index finger to index finger, and so on. But you can also invert one hand and match thumb to pinkie, index to ring finger, and so on. And there are obviously many (120, in fact) ways to tie each digit to a unique digit on the other hand.

The general idea is that a set A has as many members as a set B exactly in case there is a way to match the members of A with the members of B one-to-one. The phrase "one-to-one" may make you worry that numbers like one are being smuggled in surreptitiously. The honest way to allay this worry is to lay out the set theoretic nuts and bolts of matching one-to-one so it is clear no numbers have snuck in. Laying out these nuts and bolts is also a way of illustrating how sets have become the arena in which logic, mathematics, and more are conducted. Sets are not just the natural kind of infinity; they are also a natural kind across logic, mathematics, and beyond. Frege's aim was to reduce the mathematics of number to logic. To do so, he treated extensions (of predicates, properties, or concepts) considerably more systematically than the comparatively casual use traditional logic had made of extensions for centuries before Frege. His treatment of extensions got into enough trouble that it is at least doubtful whether the mathematics of number is reducible to logic. But Frege's systematic treatment of extensions is an important stage in sets becoming the arena of mathematics and logic.

There are two primitive predicates in our exposition of basic naïve set theory. (We'll see later what the naïveté is.) We want a predicate for the relation of, say, a senator to the set of senators. This is called the membership, or elementhood, relation, and is usually written for short as similar to the small Greek letter epsilon, or $\in$. So if S is the set of states and A is the state of Alabama, then $A \in S$ says that Alabama is a member of the set of states. If our theory were to be a theory of nothing but sets, $\in$ could be our only primitive, and in that way set theory is the laws of the membership relation. But if we want to allow room for application to things like people and rocks that we don't think of as sets, so that we can have the set of people and the set of rocks, then we should also take identity as a primitive. We write this, as usual, as the double bar, =. To say that 7 + 5 = 12 means that the number 12 is the same thing as the sum of 7 and 5.

It is central to sets that they are identical when they have the same members. There aren't two totalities of all and only the shoes. The principle that sets with the same members are identical is called extensionality. Were we discussing nothing but sets, we could take membership as our only primitive predicate and use extensionality to introduce identity. But if we include things like the Rock of Gibraltar and Peter Abelard that presumably are not sets, then since only sets have members, Gibraltar and Abelard will have the same members, namely none, and yet not be identical. So when sets are applied, it is natural to assume identity as well as membership. Extensionality distinguishes sets from predicates and properties. Two predicates like "is directly over Big Ben" and "is directly above Big Ben" are true of all and only the same things, and yet are different predicates. Being spelled the same suffices for predicates to be identical. Properties can be had by the same things and yet differ. Easy instances are empty properties like being a centaur and being a griffin. But instantiated properties, like having a heart and having a liver, also seem to differ even if all and only the animals with hearts have livers. Some say that necessary coextensiveness suffices for property identity; others reply that necessity is unclear (without making it clear what clarity requires). Whatever the rights and wrongs of that dispute, there is more worked-out and settled lore about sets than about properties, so logicians and mathematicians favor sets over properties.

There are two systematic ways to name sets. If a set is finite and we have names for its members, then curly brackets enclosing a list of those names separated by commas is a name for that set. So

{Mercury, Venus, Earth}

is the set of the three inmost planets of our solar system. Since Cantor's quarry was the infinite, such names would not have satisfied him. Suppose we have a predicate like "Ralph gave x a present." Here the variable "x" marks a blank that may be filled by singular terms (proper names, definite descriptions, demonstratives) denoting things that are or are not targets of Ralph's generosity. Abbreviate this predicate as "Px." Then

$$\{x \mid Px\}$$

is read "the set of all things, say x is one, such that Px" and in our case would be the set of all and only the recipients of Ralph's generosity. This set is called the extension of the predicate "Px," uniqueness here being justified by extensionality. The assumption that every predicate has a set that is its extension is called comprehension. Naïve set theory is the theory whose axioms are extensionality and comprehension, and as we shall see, comprehension is thought to be its naïveté.

The notation {x | Px} is called set abstraction. List terms can be replaced by abstracts on the model of

$$\{x \mid x = \text{Mercury or } x = \text{Venus or } x = \text{Earth}\},$$

so we can make do with abstraction if we wish to be economical. The abstraction notation was introduced by Giuseppe Peano. Like the definite description operator, it applies to predicates and yields singular terms. Such terms may occur in yet further predicates, whence intricate nesting may ensue. Abstraction and membership are like inverses of each other. When Pa, the predication factors into a being a member of the set of Ps; Quine calls this the principle of abstraction. When a is a member of the set of Ps, membership and abstraction cancel out, and so Pa; this Quine calls concretion.

Comprehension says there is a set of all those things not identical with themselves (or a set of all unicorns), and extensionality says it is unique. This set is called the empty set, and it is denoted by ∅, which is not the Greek letter phi, but similar to the Danish and Norwegian slashed O. Some people who think of sets as somehow constituted out of, or dependent for their existence on, their elements have metaphysical qualms about the empty set. But an empty set need be no more troubling than an empty glass. Extensionality says that a set's members suffice to fix its identity, but this is neither to say the set is constituted from its members nor to say it depends for its existence on them. Besides, the hypothesis that there is an empty set has proved its utility time and again, and confirmation need not be cowed by metaphysical intuitions.

For any objects a and b, there is a unique set {a, b} whose members are a and b. Since {a, b} and {b, a} have the same members, extensionality says they are identical. So {a, b} is called the unordered pair of a and b. When a is b, their unordered pair is the set whose sole member is a; this is written {a} and is called the unit set or singleton of a. If a is itself a set with none or many members, it will not have the same members as its singleton, so in general a should be distinguished from its singleton. (But in what might seem an excess of economic zeal, Quine favored identifying a non-set with its unit set, as he showed how to do consistently.)

The empty set and unordered pairs assure us some sets outright. There are also operations on sets that assure us their values given their arguments. The Boolean operations, named for George Boole, correspond to truth functions. Thus the union of a and b, written a ∪ b, is the set of all x such that x ∈ a or x ∈ b. (The notation "∪" is Peano's.) With ∅ and unit sets, repeated union gives us all finite sets. The intersection of a and b, written a ∩ b, is the set of all x such that x ∈ a and x ∈ b. (The notation "∩" is also Peano's.) The intersection of the set of all odd numbers and the set of all even numbers is the empty set. Such sets are called disjoint. Without the empty set, disjoint sets would have no intersection, and we could not form a ∩ b without checking that a and b meet; the convenience of always being able to form a ∩ b is an example of the utility of ∅. The complement of a, written variously whence we pick ā, is the set of all things not in a. Complements, as we shall see, are a mark of naïveté, and sophistication sometimes favors differences, written a – b and explained as the set of all x such that x ∈ a but x ∉ b. (The ∈ with a stroke is denial of membership.)

In addition to Boolean operations, we also have the subset, or inclusion, relation. A set a is a subset of a set b, written a ⊆ b (like a softened less-or-equal sign), just in case all members of a are members of b. If b is also a subset of a, then they have the same members and so are identical. Note that when a ∈ b, then every member of {a} is a member of b, so {a} ⊆ b. Thereby may, but need not, hang a tale. Some people picture a layered world. On the ground floor, or layer 0, are the non-sets, the shoes and ships and so on. On layer 1 are the sets of things on layer 0. On layer 2 are either the sets of things on layer 1 (if, like Russell, you like your layers exclusive) or the things on layer 0 or layer 1 (if you like your layers cumulative). And so on for longer than you might expect. On this picture, Plato and everybody else is on layer 0, while the set of people and Plato's unit set are on layer 1. Then ∈ relates *across* layers (Plato is a member of his unit set and the set of people), while ⊆ relates *within* layers (the singleton of Plato is a subset of the set of people). It took a long, long time for us to learn to distinguish between ∈ and ⊆. The distinction was drawn clearly and driven home only in the nineteenth century. The premisses and conclusions of traditional syllogisms were either universal (All men are mortal) or particular (Some dogs are terriers). Singular premisses (Socrates is a man) were recognized, as in the old textbook inference from our universal and singular premisses to a singular conclusion (Socrates is mortal), but the effort to assimilate the singular to the universal or particular encouraged a confusion between ∈ and ⊆, as if Socrates were a tiny species.

The picture of layers helps distinguish between ∈ and ⊆, which is a virtue of it. Some people think it is the only right, or possible or coherent, way to picture the world. Maybe, but that view carries substantial commitments, so be wary of buying into it thoughtlessly. We will see larger issues later, but here is a smaller one. Consider propositions and self-reference. Russell (and probably Leibniz) thought of propositions as extensions of sentences as sets are extensions of predicates and as its denotation is the extension of a name. For example, the proposition that Socrates is bald would be the ordered pair <s, B> whose first member, s, is Socrates and whose second member, B, is the set of bald people. (We will get to ordered pairs very soon, but for now the important thing is that when a is different from b, the ordered pair <a, b> with a first and b second is a different thing from <b, a> with b first and a second.) This proposition <s, B> is true just in case s ∈ B, which opens a natural story about truth. Now consider a self-referential proposition like

This proposition can be expressed in eight words.

Let E be the set of propositions expressible in eight words, and let p be the proposition we are now considering. On Russell's conception, p is <p, E>, the doubling being the self-reference. We will soon construe ordered pairs as sets, and on the layered picture, an ordered pair will lie two layers above its members. On a layered picture, a set lies on a layer higher than its members, which would forbid self-referential propositions. But proposition p seems in order, indeed true, and we will later see more systematic reasons for reluctance to give up self-reference. It would not be shrewd to commit fully to the layered picture unreflectively, even if it is the conventional wisdom.

The set of tigers is the extension of the predicate "is a tiger." This predicate is unary (Latin) or monadic (Greek), both of which mean that it has one blank or empty space that on being filled with a singular term (like "Tony") yields a sentence. Each predicate has a number of blanks, filling all of which with singular terms yields a sentence. This number is called the predicate's polyadicity (Greek) or, much more rarely, its arity (Latin). The predicate "love" is binary (or dyadic) since it has two blanks for names, as in "Regina loved Søren," and "give" is ternary (or triadic) since it has three blanks, as in "The president gave the contract to his brother-in-law." The Greek and Latin of logicians give out and they speak instead of 5-adic (or 5-ary) predicates. (Some predicates, as in "Andrew united Bob, Curt, David, and Ed in a conspiracy," seem to lack a unique polyadicity, but they are rare.) In reckoning the polyadicity of a predicate in a sentence, one may count as many of the occurrences of singular terms as one wishes. For example, in

Richard gave the diamond to Elizabeth

one may count three singular terms filling the blanks in a ternary predicate, but one may count any two filling blanks in a more complex binary predicate, and one may count any one filling the blank in a yet more complex unary predicate. The logician is prescinding from grammatical roles (like direct or indirect object) and, as it were, counting several singular terms all as several subjects of a polyadic predicate.

The set of tigers is the extension of the monadic predicate "is a tiger." We would also like extensions for polyadic predicates. As tigers one by one fill out the extension of "is a tiger," we expect pairs to fill out the extension of a dynadic predicate like "loves." But we notice immediately that order matters. Regina seems to have been a normal person and to have loved Søren, but we owe Kierkegaard's works at least in part to his inability to make up his mind that he loved Regina. Unrequited love shows that the members of the extension of "loves" cannot be unordered pairs. We write ordered pairs with angle brackets, so the ordered pair whose first member is Regina and whose second is Søren is <r, s>. This pair is in the extension of "loves," but <s, r> is not, so it had better turn out that <s, r> ≠ <r, s>. This illustrates a central aspect of order: when a ≠ b, <a, b> ≠ <b, a>; order alone suffices to distinguish ordered pairs. More generally, <a, b> = <c, d> if and only if a = c and b = d (while, by contrast, if a = d and b = c, then {a, b} = {c, d}). This principle articulates what Tarski in the 1920s will call a material adequacy condition, that is, a condition an account of something (in Tarski's case truth, in ours, order) should meet to be adequate. In the 1910s, Norbert Wiener and Kazimierz Kuratowski each showed a way to explain the ordered pair in the primitive terms of set theory so as to satisfy the adequacy condition. (Quine said this work is a philosophical paradigm.)

We mostly follow Kuratowski, whose later account explains <a, b> as {{a}, {a, b}}. It would be a mistake to stare at this hoping for insight into order. Such insight as there is to be had was already articulated in the adequacy condition. Kuratowski's account is adequate (as is Wiener's different one) if it proves to satisfy the adequacy condition.

There is no enlightenment to be found in the proof that Kuratowski's account works, but students always ask to see a proof, so here goes. Suppose <a, b> = <c, d>. Then, by Kuratowski's account, {{a}, {a, b}} = {{c}, {c, d}}. Since {a} is a member of the set on the left, it is in the one on the right, so it is {c} or {c, d}. In the first case, a is c, while in the second both c and d are a. So in any case, a = c. Next we distinguish two cases. For the first, suppose a = b. Since {c, d} is in the second set, it is in the first, so it is {a} or {a, b}; if it is {a}, then d is a, which is b; while if it is {a, b}, then d is a or b, which are identical, so d is again b. Hence if a = b, b = d. So for our second case, suppose a ≠ b. If b = c, then since a = c, a = b, so since we're supposing a ≠ b, b ≠ c. Then b ∈ {c, d}, so since b ≠ c, b = d. Hence, in any case, b = d, as we were to show. This argument is a welter of unmemorable cases, so don't worry if your attention glazed over; what matters is that it works. Russell called Kuratowski's (and Wiener's) construction a trick.

Once we have ordered pairs, we may take an ordered triple <a, b, c> as <<a, b>, c>, an ordered pair whose first member is an ordered pair. An ordered quadruple <a, b, c, d> is <<a, b, c>, d>, and so on through all the ordered n-tuples. Then we may take the extension of an n-adic predicate to be a set of ordered n-tuples. We should work an example to fix ideas. The extension of the binary predicate "a is n years old at noon today" (where the blanks in the predicate are marked with the variables "a" and "n") is the set of ordered pairs <a, n> such that a is n years old at noon today. This one could also think of as (the noon today time slice of) the age relation. Let us focus on people: let P be the set of people (alive at noon today) and let N = {0, 1,…} be the set of all natural (i.e., non-negative, whole) numbers. The set of all ordered pairs <a, b> whose first member a is an element of P and whose second member b is an element of N is called the Cartesian or cross product of P and N. It is written P × N. It is called Cartesian in memory of rectangular Cartesian coordinates for the points on a Euclidian plane; it is called cross because if A has n members and B has k members, then A × B has n times k members (which hints at reconstructing arithmetic in set theory). A binary relation between people (alive at noon today) and natural numbers is any old subset of P × N. A relation between members of A and members of B is a subset of A × B. Age is a relation between people and numbers; age (at noon today) is

{<p, n> | p ∈ P and n ∈ N and p is n years old at noon today}, which is a subset of $P \times N$. An n-ary relation among members of n sets $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \ldots \times A_n$, the set of ordered n-tuples whose first member is in A_1, whose second is in $A_2, \ldots$, and whose n^{th} is in A_n.

Of course, a relation may hold between some members of a set like P and others. Such a relation is a subset of $P \times P$. For example, parenthood (at noon today) is a subset of the set of all ordered pairs <a, b> of people (alive at noon today). We use exponential notation for cross products whose factors are identical: P^2 is $P \times P$; P^3 is $P \times P \times P$; and so on. Let R be the parenthood relation just mentioned. Let W be the set of all women (alive at noon today). We might want to restrict the first members of a relation S to elements of a set A; we would write $S \upharpoonleft A$ for the set of pairs <x, y> in S such that $x \in A$. Then the motherhood relation is $R \upharpoonleft W$. To restrict the second members of pairs in S to A, we write $S \upharpoonright A$. Then the daughterhood relation is $R \upharpoonright W$. To restrict both to A, we write $S \uparrow A$, so $R \uparrow W$ is the mother–daughter relation.

Aristotle's syllogistic logic is geared for unary predicates. It had long been recognized that there are arguments whose conclusions clearly follow from their premisses but where syllogistic cannot certify these arguments because the arguments' success turns on polyadic predicates. Here is an example from Augustus De Morgan in the nineteenth century:

All horses are animals.
Hence, all heads of horses are heads of animals,

where the dyadic predicate "x is a head of y" is crucial. It was not until the nineteenth century that a systematic account of relations began.

In addition to De Morgan, Charles Sanders Peirce and Ernst Schröder were central in the articulation of relations. Notation like $R \uparrow W$ is just one fruit of their work. The fact that ordered pairs were worked out set theoretically pretty much at the end of the articulation of relations shows how hard it was to command a clear view of relations.

In the seventeenth century, Newton and Leibniz focused our attention on functions. The path of a particle in, for simplicity, the plane rather than space is a continuous curve, and using Cartesian coordinates the ordinates of points along the curve can often be given as mathematical functions of the abscissae. The speed of this particle at a point along its path will be given by the derivative of such a function, and conversely, the path is given by the integral of the particle's velocity; anyone who

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Excerpt
More information

has done some calculus knows that differentiation and integration are the meat and potatoes of Newton's and Leibniz's calculus. At school we were all programmed in algorithms for computing the sums and products of natural numbers, and addition and multiplication are also functions. Such education inclines us to think of functions in terms of ways of calculating the output, or value, of a function at its inputs, or arguments. A somewhat less intentional image of a function pictures it as a bunch of arrows, one *from* each argument *to* the value of the function for that argument; the collection of its arguments is called the function's domain, while the collection in which its values lie is called the function's range, so on this picture a function is a collection of arrows arcing from its domain into its range.

As late as Kant at the end of the eighteenth century, curves were the leading image of functions. Through the nineteenth century, people worked out an extensional conception of a function. The calculus is infinitary, and the geometrical imagination trusted since Euclid began to go awry in the infinities of the calculus. Much of nineteenth-century mathematics was given over to a process called the arithmetization of analysis, which is what calculus grows up into. The aim of this process is to replace geometry, especially in analysis, with the mathematics of number, or later, set theory. An extensional conception of a function arises by starting from the picture of a bunch of arrows arcing from its domain to its range, and then discarding everything except the ordered pairs whose first members are the arguments and whose second are the values; only input and output remain, and we don't worry about how what goes in becomes what comes out.

We write $f : A \to B$ to mean that f is a function whose domain is a set A and whose range is a set B. But we have just seen that on the extensional conception this means that f is a set of ordered pairs whose first members lie in A and whose second lie in B; that is, it is a relation between members of A and B. There are two special conditions that such a relation must meet in order to be a function. First, for every member a of A, there is at least one member b of B such that <a, b> is in f, that is, f relates a to b. Second, for each a in A, there is at most one b in B such that <a, b> is in f. In the crochets of logic, the first condition is that

$$(\forall a)(a \in A \to (\exists b)(b \in B \wedge \langle a, b\rangle \in f)),$$

while the second is that

$$(\forall a)(a \in A \to (\forall b)(\forall c)((b \in B \wedge c \in B \wedge \langle a, b\rangle \in f \wedge \langle a, c\rangle \in f) \to a = c)),$$