Triangulated categories: definitions, properties, and examples

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Triangulated categories were introduced in the mid 1960's by J.L. Verdier in his thesis, reprinted in [16]. Axioms similar to Verdier's were independently also suggested in [2]. Having their origins in algebraic geometry and algebraic topology, triangulated categories have by now become indispensable in many different areas of mathematics. Although the axioms might seem a bit opaque at first sight it turned out that very many different objects actually do carry a triangulated structure. Nowadays there are important applications of triangulated categories in areas like algebraic geometry (derived categories of coherent sheaves, theory of motives) algebraic topology (stable homotopy theory), commutative algebra, differential geometry (Fukaya categories), microlocal analysis or representation theory (derived and stable module categories).

It seems that the importance of triangulated categories in modern mathematics is growing even further in recent years, with many new applications only recently found; see B. Keller's article in this volume for one striking example, namely the cluster categories occurring in the context of S. Fomin and A. Zelevinsky's cluster algebras which have been introduced only around 2000.

In this chapter we aim at setting the scene for the survey articles in this volume by providing the relevant basic definitions, deducing some elementary general properties of triangulated categories, and providing a few examples.

Certainly, this cannot be a comprehensive introduction to the subject. For more details we refer to one of the well-written textbooks on triangulated categories, e.g. [4], [5], [7], [8], [12], [17], and for further topics also to the surveys in this volume.

This introductory chapter should be accessible for a reader with a good background in algebra and some basic knowledge of category theory and homological algebra.

1. Additive categories

In this first section we shall discuss the fundamental notion of an additive category and provide some examples. In particular, the category of complexes over an additive category is introduced which will play a fundamental role in the sequel.

Definition 1.1. *A category $\mathcal{A}$ is called an* additive category *if the following conditions hold:*

(A1) *For every pair of objects X, Y the set of morphisms* $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ *is an abelian group and the composition of morphisms*

$$\mathrm{Hom}_{\mathcal{A}}(Y, Z) \times \mathrm{Hom}_{\mathcal{A}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(X, Z)$$

is bilinear over the integers.

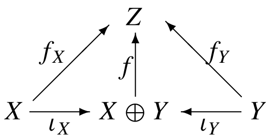
(A2) *$\mathcal{A}$ contains a zero object* 0 *(i.e. for every object X in $\mathcal{A}$ each morphism set* $\mathrm{Hom}_{\mathcal{A}}(X, 0)$ *and* $\mathrm{Hom}_{\mathcal{A}}(0, X)$ *has precisely one element).*

(A3) *For every pair of objects X, Y in $\mathcal{A}$ there exists a coproduct $X \oplus Y$ in $\mathcal{A}$.*

Remark 1.2.

(i) A category satisfying $(A1)$ and $(A2)$ is called a *preadditive* category.

(ii) We recall the notion of coproduct from category theory. Let $\mathcal{C}$ be a category and X, Y objects in $\mathcal{C}$. A coproduct of X and Y in $\mathcal{C}$ is an object $X \oplus Y$ together with morphisms $\iota_X : X \to X \oplus Y$ and $\iota_Y : Y \to X \oplus Y$ satisfying the following universal property: for every object Z in $\mathcal{C}$ and morphisms $f_X : X \to Z$ and $f_Y : Y \to Z$ there is a unique morphism $f : X \oplus Y \to Z$ making the following diagram commutative.



Example 1.3.

(i) Let R be a ring and consider R as a category $\mathcal{C}_R$ with only one object. The unique morphism set is the underlying abelian group and composition of morphisms is given by ring multiplication. Then $\mathcal{C}_R$ satisfies (A1) and (A2), thus preadditive categories can be seen as generalizations of rings. But $\mathcal{C}_R$ is not additive in general; in fact the coproduct of the unique object with itself would have to be again this object together with fixed ring elements ι_1, ι_2, and the universal property would mean that for arbitrary ring elements f_1, f_2 there existed a unique element f factoring them as $f_1 = f\iota_1$ and $f_2 = f\iota_2$.

(ii) Let R be a ring (associative, with unit element). Then the category **R-Mod** of all R-modules is additive. Similarly, the category **R-mod** of finitely generated R-modules is additive. In particular, the categories **Ab** of abelian groups and $\mathbf{Vec_K}$ of vector spaces over a field K are additive.

(iii) The full subcategory of **Ab** of free abelian groups is additive.

(iv) For a ring R the full subcategory **R-Proj** of projective R-modules is additive; similarly for **R-proj**, the category of finitely generated projective R-modules.

1.1. The category of complexes

Let $\mathcal{A}$ be an additive category. A *complex over* $\mathcal{A}$ is a family $X = (X_n, d_n^X)_{n\in\mathbb{Z}}$ where X_n are objects in $\mathcal{A}$ and $d_n^X : X_n \to X_{n-1}$ are morphisms such that $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Usually, a complex is written as a sequence of objects and morphisms as follows.

$$\ldots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \ldots$$

Let $X = (X_n, d_n^X)$ and $Y = (Y_n, d_n^Y)$ be complexes over $\mathcal{A}$. A *morphism of complexes* $f : X \to Y$ is a family of morphisms $f = (f_n : X_n \to Y_n)_{n\in\mathbb{Z}}$ satisfying $d_n^Y \circ f_n = f_{n-1} \circ d_n^X$ for all $n \in \mathbb{Z}$, i.e. we have the following commutative diagram.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{n+1} & \longrightarrow & X_n & \longrightarrow & X_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & Y_{n+1} & \longrightarrow & Y_n & \longrightarrow & Y_{n-1} & \longrightarrow & \cdots
\end{array}$$

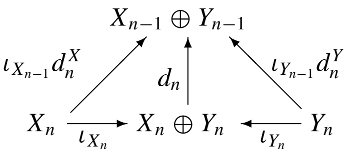
The complexes over an additive category $\mathcal{A}$ together with the morphisms of complexes form a category $\mathbf{C}(\mathcal{A})$, the *category of complexes over* $\mathcal{A}$.

Proposition 1.4. *Let $\mathcal{A}$ be an additive category. Then the category of complexes $\mathbf{C}(\mathcal{A})$ is again additive.*

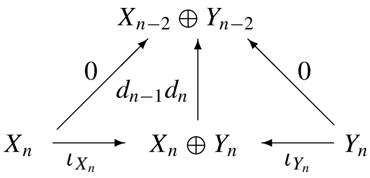
Proof. (A1) Addition of morphisms is defined degreewise, i.e. for two morphisms $f = (f_n)_{n\in\mathbb{Z}}$ and $g = (g_n)_{n\in\mathbb{Z}}$ from X to Y their sum is $f + g := (f_n + g_n)_{n\in\mathbb{Z}}$. Using the additive structure of $\mathcal{A}$ it is then easy to check that (A1) holds.

(A2) The zero object in $\mathbf{C}(\mathcal{A})$ is the complex $(0_\mathcal{A}, d)$ where $0_\mathcal{A}$ is the zero object of the additive category $\mathcal{A}$ and all differentials are the unique (zero) morphism on the zero object.

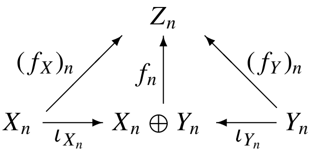
(A3) The coproduct of two complexes $X = (X_n, d_n^X)$ and $Y = (Y_n, d_n^Y)$ is defined degreewise by using the coproduct in the additive category $\mathcal{A}$. More precisely $X \oplus Y = (X_n \oplus Y_n, d_n)_{n\in\mathbb{Z}}$ where the differential is obtained by the universal property as in the following diagram.



From uniqueness in the universal property applied to



it follows that $d_{n-1} \circ d_n = 0$. This complex indeed satisfies the properties of a coproduct in the category of complexes $\mathbf{C}(\mathcal{A})$, with morphisms of complexes $\iota_X = (\iota_{X_n})_{n\in\mathbb{Z}} : X \to X \oplus Y$ and $\iota_Y = (\iota_{Y_n})_{n\in\mathbb{Z}} : Y \to X \oplus Y$. For checking the universal property let Z be an arbitrary complex and let $f_X : X \to Z$ and $f_Y : Y \to Z$ be arbitrary morphisms. The unique morphism of complexes satisfying $f_X = f \circ \iota_X$ and $f_Y = f \circ \iota_Y$ is $f = (f_n)_{n\in\mathbb{Z}} : X \oplus Y \to Z$, where f_n is obtained from the universal property in degree n as in the following diagram.



□

Remark 1.5. For complexes over $\mathcal{A} = $ **R-Mod** where R is a ring with unit (and other similar examples) the coproduct of two complexes is more easily be described on elements as $X \oplus Y = (X_n \oplus Y_n, d_n)_{n\in\mathbb{Z}}$ where the differential is given by $d_n(x_n, y_n) = (d_n^X(x_n), d_n^Y(y_n))$ for $x_n \in X_n$ and $y_n \in Y_n$, and with morphisms $\iota_X : X \to X \oplus Y$ and $\iota_Y : Y \to X \oplus Y$ being the inclusion maps. The unique morphism of complexes satisfying $f_X = f \circ \iota_X$ and $f_Y = f \circ \iota_Y$ is then given by $f_n(x_n, y_n) = f_X(x_n) + f_Y(y_n)$.

1.2. The homotopy category of complexes

Let $\mathcal{A}$ be an additive category. Morphisms $f, g : X \to Y$ in the category $\mathbf{C}(\mathcal{A})$ of complexes are called *homotopic*, denoted $f \sim g$, if there exists a family $(s_n)_{n\in\mathbb{Z}}$ of morphisms $s_n : X_n \to Y_{n+1}$ in $\mathcal{A}$, satisfying $f_n - g_n = d^Y_{n+1}s_n + s_{n-1}d^X_n$ for all $n \in \mathbb{Z}$.

In particular, setting g to be the zero morphism, we can speak of morphisms being *homotopic to zero*.

It is easy to check that $\sim$ is an equivalence relation. Moreover, if $f \sim g : X \to Y$ are homotopic and $\alpha : W \to X$ is an arbitrary morphism of complexes, then also the compositions $f\alpha \sim g\alpha$ are homotopic. In fact, $(s_n\alpha_n)_{n\in\mathbb{Z}}$ are homotopy maps since

$$(f_n - g_n)\alpha_n = (d^Y_{n+1}s_n + s_{n-1}d^X_n)\alpha_n = d^Y_{n+1}(s_n\alpha_n) + (s_{n-1}\alpha_{n-1})d^W_n.$$

Similarly, if $f, g : X \to Y$ are homotopic and $\beta : Y \to Z$ is a morphism of complexes then $\beta f \sim \beta g$ are homotopic.

This implies that we have a well-defined composition of equivalence classes of morphisms modulo homotopy by defining the composition on representatives.

Definition 1.6. *Let $\mathcal{A}$ be an additive category. The* homotopy category $\mathbf{K}(\mathcal{A})$ *has the same objects as the category $\mathbf{C}(\mathcal{A})$ of complexes over $\mathcal{A}$. The morphisms in the homotopy category are the equivalence classes of morphisms in $\mathbf{C}(\mathcal{A})$ modulo homotopy, i.e.*

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, Y) := \mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X, Y)/\sim .$$

Proposition 1.7. *Let $\mathcal{A}$ be an additive category. Then the homotopy category $\mathbf{K}(\mathcal{A})$ is again an additive category.*

Proof. Addition of morphisms in $\mathbf{K}(\mathcal{A})$ is defined via addition on representatives (it is an easy observation that this is well-defined) and then the sets of morphisms $\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, Y)$ inherit the structure of an abelian group from the category $\mathbf{C}(\mathcal{A})$ of complexes, and also bilinearity of composition. Moreover, the zero object is the same as in $\mathbf{C}(\mathcal{A})$.

It remains to be checked that the universal property of the coproduct $X \oplus Y$ in $\mathbf{C}(\mathcal{A})$ (cf. Proposition 1.4) also carries over to the homotopy category. In fact, the equivalence classes of the morphisms ι_X, ι_Y and f still make the relevant diagram (cf. Remark 1.2) commutative; for uniqueness we observe that if there is another morphism g making the diagram for the universal property commutative in $\mathbf{K}(\mathcal{A})$, i.e. up to homotopy, then this gives a homotopy between f and g. □

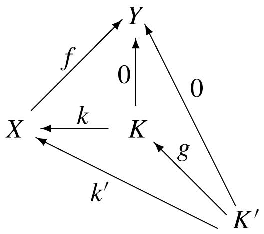
2. Abelian categories

In this section we shall review the fundamental definition of an abelian category, including the necessary background on the categorical notions of kernels and cokernels. The prototype example of an abelian category will be the category **R-Mod** of modules over a ring R; but we will also see other examples in due course.

We first recall some notions from category theory. Let $\mathcal{A}$ be an additive category; in particular for every pair of objects X, Y there is a zero morphism, namely the composition of the unique morphisms $X \to 0 \to Y$ involving the zero object of $\mathcal{A}$.

The *kernel* of a morphism $f : X \to Y$ is an object K together with a morphism $k : K \to X$ such that

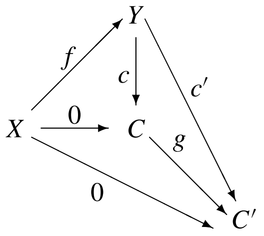
(i) $f \circ k = 0$,
(ii) (universal property) for every morphism $k' : K' \to X$ such that $f \circ k' = 0$, there is a unique morphism $g : K' \to K$ making the following diagram commutative.



By the usual universal property argument, the kernel, if it exists, is unique up to isomorphism; notation: ker f.

Dually, the *cokernel* of a morphism $f : X \to Y$ is an object C together with a morphism $c : Y \to C$ such that

(i) $c \circ f = 0$,
(ii) (universal property) for every morphism $c' : Y \to C'$ such that $c' \circ f = 0$, there is a unique morphism $g : C \to C'$ making the following diagram commutative.



Again, the cokernel, if it exists, is unique up to isomorphism; notation: coker f.

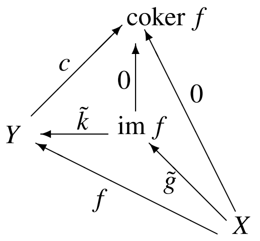
If the above morphism $k : \ker f \to X$ has a cokernel in $\mathcal{A}$, this is called the *coimage* of f, and it is denoted by coim f.

If the above morphism $c : Y \to \operatorname{coker} f$ has a kernel in $\mathcal{A}$, this is called the *image* of f and it is denoted by im f.

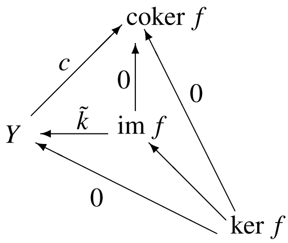
Example 2.1. Let R be a ring. In the category **R-Mod** of all R-modules the categorical kernels and cokernels are the usual ones, i.e., for a morphism $f : X \to Y$ we have $\ker f = \{x \in X \mid f(x) = 0\}$ and $\operatorname{coker} f = Y/\operatorname{im} f$ where $\operatorname{im} f = \{f(x) \mid x \in X\}$ is the usual image of f.

Remark 2.2. Suppose that for a morphism f both the coimage and the image exist. Then we claim that it follows from the universal properties that there is a natural morphism coim $f \to$ im f.

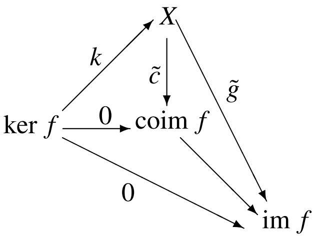
In fact, the image of f is the kernel of $c : Y \to \operatorname{coker} f$, hence there is a morphism $\tilde{k} : \operatorname{im} f \to Y$ such that $c \circ \tilde{k} = 0$ and by the universal property there exists a unique morphism $\tilde{g} : X \to \operatorname{im} f$ making the following diagram commutative.



Note that $\tilde{k} \circ \tilde{g} \circ k = f \circ k = 0$, which implies that $\tilde{g} \circ k : \ker f \to \operatorname{im} f$ must be zero, by using the uniqueness in the following diagram.



Then we can consider the following diagram for the universal property of the coimage



and deduce that there is a unique morphism coim $f \to$ im f, as desired.

Definition 2.3. *An additive category $\mathcal{A}$ is called an* abelian category *if the following axioms are satisfied:*

(A4) *Every morphism in $\mathcal{A}$ has a kernel and a cokernel.*
(A5) *For every morphism $f : X \to Y$ in $\mathcal{A}$, the natural morphism* coim $f \to$ im f *is an isomorphism.*

Example 2.4.

(i) Let R be a ring. The category **R-Mod** of all R-modules is an abelian category. In fact, (A5) follows directly from the isomorphism theorem for R-modules.

However, the subcategory **R-mod** of finitely generated modules is not abelian in general since kernels of homomorphisms between finitely generated modules need not be finitely generated. Indeed we have that **R-mod** is an abelian category if and only if R is Noetherian.

In particular, the category of finite-dimensional vector spaces over a field is abelian, and the category of finitely generated abelian groups is abelian.

(ii) The subcategory of **Ab** consisting of free abelian groups is not abelian.

On the other hand, for a prime number p, the abelian p-groups form an abelian subcategory of **Ab** (an abelian group is called a p-group if for every element a we have $p^k a = 0$ for some k).

(iii) For finding examples of additive categories satisfying (A4) but failing to be abelian, the following observation can be useful. Suppose $f : X \to Y$ is a morphism with ker $f = 0$ and coker $f = 0$, i.e. a monomorphism and an epimorphism. Then the coimage of f is the identity on X, the image of f is the identity on Y and hence the natural morphism coim $f \to$ im f is just f itself. So in this special case the axiom (A5) states that a morphism which is a monomorphism and an epimorphism must be invertible.

(iv) Explicit examples of additive categories where axiom (A5) fails for the above reason are the category of topological abelian groups (with continuous group homomorphisms) or the category of Banach complex vector spaces (with continuous linear maps). In such categories the cokernel of a morphism $f : X \to Y$ is of the form $Y/\overline{\mathrm{im}}_f$ where $\overline{\mathrm{im}}_f$ is the *closure* of the usual set-theoretic image of f. In particular, the natural morphism $\mathrm{coim}\, f \to \mathrm{im}\, f$ is the inclusion of the usual image of f into its closure, and this is in general not an isomorphism.

Proposition 2.5. *Let $\mathcal{A}$ be an abelian category. Then the category of complexes $\mathbf{C}(\mathcal{A})$ is also abelian.*

Proof. We have seen in Proposition 1.4 that $\mathbf{C}(\mathcal{A})$ is an additive category, so it remains to verify the axioms (A4) and (A5).

(A4) Let $f : X \to Y$ be a morphism in $\mathbf{C}(\mathcal{A})$, i.e. $f = (f_n)_{n\in\mathbb{Z}}$ with $f_n : X_n \to Y_n$ morphisms in $\mathcal{A}$. We show the existence of a kernel and leave the details of the dual argument for the cokernel as an exercise.

Since $\mathcal{A}$ is abelian, each morphism $f_n : X_n \to Y_n$ has a kernel $K_n := \ker f_n$ in $\mathcal{A}$, coming with a morphism $k_n : K_n \to X_n$ satisfying the above universal property. Note that for every $n \in \mathbb{Z}$ we have $f_{n-1} \circ d_n^X \circ k_n = d_n^Y \circ f_n \circ k_n = 0$. Then it follows by the universal property of kernels that there is a unique morphism $d_n^K : K_n \to K_{n-1}$ such that $k_{n-1} \circ d_n^K = d_n^X \circ k_n$. Note that

$$k_{n-1} \circ d_n^K \circ d_{n+1}^K = d_n^X \circ k_n \circ d_{n+1}^K = d_n^X \circ d_{n+1}^X \circ k_{n+1} = 0$$

since X is a complex. By uniqueness of the map in the universal property of K_{n-1} it follows that $d_n^K \circ d_{n+1}^K = 0$, i.e. (K_n, d_n^K) is a complex.

Combining the universal properties of the kernels K_n it easily follows that the complex (K_n, d_n^K) indeed satisfies the universal property for the kernel of f in $\mathbf{C}(\mathcal{A})$.

(A5) The crucial observation is that a morphism of complexes $f = (f_n) : X \to Y$ is an isomorphism in $\mathbf{C}(\mathcal{A})$ if and only if each f_n is an isomorphism in $\mathcal{A}$. In fact, if each f_n is an isomorphism, with inverse g_n, then the family $g = (g_n)$ is automatically a morphism of complexes (and hence clearly an inverse to f in $\mathbf{C}(\mathcal{A})$): for all $n \in \mathbb{Z}$ we have

$$d_{n+1}^X \circ g_{n+1} = g_n \circ f_n \circ d_{n+1}^X \circ g_{n+1} = g_n \circ d_{n+1}^Y \circ f_{n+1} \circ g_{n+1} = g_n \circ d_{n+1}^Y.$$

The reverse implication is obvious.

For axiom (A5) now consider the natural morphism $\mathrm{coim}\, f \to \mathrm{im}\, f$. In the proof of (A4) above we have seen that kernels and cokernels in $\mathbf{C}(\mathcal{A})$, and hence also the morphism coim $f \to \operatorname{im} f$, are obtained degreewise. But since $\mathcal{A}$ is abelian by assumption, we know that for every n the natural morphism coim $f_n \to \operatorname{im} f_n$ in $\mathcal{A}$ is indeed an isomorphism. Then, by the introductory remark, the morphism of complexes (coim $f_n \to \operatorname{im} f_n)_{n\in\mathbb{Z}}$ is an isomorphism in $\mathbf{C}(\mathcal{A})$. □

An important observation is that the homotopy category $\mathbf{K}(\mathcal{A})$ is not abelian in general, even if $\mathcal{A}$ is abelian.

Example 2.6. We provide an explicit example for the failure of axiom (A4) in a homotopy category. Consider the abelian category $\mathcal{A} = \mathbf{Ab}$ of abelian groups.

Let $f : X \to Y$ be the following morphism of complexes of abelian groups, with non-zero entries in degrees 1 and 0,

$$\begin{array}{ccccccccc}
\cdots & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \cdots \\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & \\
\cdots & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 & \cdots
\end{array}$$

In the category $\mathbf{C}(\mathbf{Ab})$ of complexes f is non-zero and has the zero complex as kernel (cf. the proof of Proposition 2.5). However, f is homotopic to zero (with the identity as homotopy map), i.e. $f = 0$ in the homotopy category $\mathbf{K}(\mathbf{Ab})$.

We claim that in the homotopy category f has no kernel. Recall the categorical definition of the kernel of a morphism $f : X \to Y$ from Section 2.

Suppose for a contradiction that our morphism f had a kernel in $\mathbf{K}(\mathbf{Ab})$. So there is a complex $\ldots \to K_1 \to K_0 \to K_{-1} \to \ldots$ and a morphism $k = k_0 : K_0 \to \mathbb{Z}$ of abelian groups (in all other degrees the map k has to be zero since X is concentrated in degree 0). The image of k, being a subgroup of $\mathbb{Z}$, has the form $r\mathbb{Z}$ for some fixed $r \in \mathbb{Z}$. Now choose $K' = X$ and consider the morphisms $l : K' \to X$ given by multiplication with l for any $l \in \mathbb{Z}$. Clearly, $f \circ l = 0$ in $\mathbf{K}(\mathbf{Ab})$ since $f = 0$ in $\mathbf{K}(\mathbf{Ab})$. According to the universal property of a kernel, there must exist (unique) morphisms $u_l : \mathbb{Z} \to K_0$ such that $k \circ u_l = l$ up to homotopy. However, these maps are from $K' = X$ to X and this complex is concentrated in degree 0. Thus there are no non-zero homotopy maps and so $k \circ u_l = l$ as morphism of abelian groups. But the image of $k \circ u_l$ is contained in the image of k which is $r\mathbb{Z}$ for a fixed r, so $k \circ u_l = l$ can not hold for arbitrary $l \in \mathbb{Z}$, a contradiction.

Hence axiom (A4) fails and therefore the homotopy category $\mathbf{K}(\mathbf{Ab})$ is not an abelian category.