I

Basics on torus embeddings; examples

1 Torus embeddings over the complex numbers

We wish to review here quickly some results of TE I† and to give a more explicit description of the complex varieties obtained via certain real spaces of half the dimension.

Let T be an algebraic torus, i.e., $T \cong \mathbb{G}_m^n$ for some n, and let

$$M = \operatorname{Hom}(T, \mathbb{G}_m) ,$$

the character group of T, and

$$N = \operatorname{Hom}(\mathbb{G}_m, T) ,$$

the group of 'one-parameter' subgroups of T (in the algebraic sense).

Then $M \cong \mathbb{Z}^n$ and $N \cong \mathbb{Z}^n$, and there is a natural non-degenerate pairing $\langle \cdot, \cdot \rangle : M \times N \longrightarrow \mathbb{Z}$ of determinant 1. All this is valid over any field k. When $k = \mathbb{C}$, however, T can be described analytically as $\widetilde{T}/\pi$, where $\widetilde{T}$ is a complex vector space and π is a discrete subgroup, generating $\widetilde{T}$ over $\mathbb{C}$ and isomorphic to $\mathbb{Z}^n$. Here $\widetilde{T}$ is the universal covering space of T and π is $\pi_1(T)$ acting on $\widetilde{T}$ via translations. Note, however, that for all $a \in \pi$ the map

$$\begin{aligned} \widetilde{\phi}_a : \mathbb{C} &\longrightarrow \widetilde{T} \\ \lambda &\longmapsto \lambda \cdot a \end{aligned}$$

induces a map

$$\phi_a : \mathbb{C}/\mathbb{Z} \longrightarrow \widetilde{T}/\pi = T ,$$

and that $\mathbb{C}/\mathbb{Z} \cong \mathbb{G}_m$ canonically via $\lambda \longmapsto \mathrm{e}^{2\pi i \lambda}$. Thus every $a \in \pi$ induces $\phi_a \in N$, and this is easily checked to be an isomorphism between π and N. Thus π is just N up to a canonical identification. Since $\widetilde{T} = \pi \otimes \mathbb{C}$, it follows that we have canonical maps:

† Recall this reference from p. x.

(i) $N \cong$ the usual topological π_1 of T;

(ii) $N \otimes \mathbb{C} \cong$ the universal covering space of T;

(iii) $(N \otimes \mathbb{C})/N \cong T$.

We abbreviate $N \otimes \mathbb{C}$ by $N_{\mathbb{C}}$ and $N \otimes \mathbb{R}$ by $N_{\mathbb{R}}$.

Next, in the isomorphism $N_{\mathbb{C}}/N \cong T$, consider the subgroup corresponding to $N_{\mathbb{R}}/N$: this a compact real torus, and is the maximal compact subgroup of T. We denote it by T_c (short for "compact torus"). Moreover, $N_{\mathbb{R}} \subset N_{\mathbb{C}}$ has a natural complement, viz. $\mathrm{i}N_{\mathbb{R}}$, and, by quotienting, $\mathrm{i}N_{\mathbb{R}}$ injects into $N_{\mathbb{C}}/N$. In other words, we get a canonical decomposition

$$N_{\mathbb{C}}/N \cong (N_{\mathbb{R}}/N) \times (\mathrm{i}N_{\mathbb{R}}) ,$$

and hence (dividing by i in the second factor)

$$T \cong T_c \times N_{\mathbb{R}} .$$

We denote the projection $T \longrightarrow N_{\mathbb{R}}$ by "ord," which is then defined by

$$\mathrm{ord}(x + \mathrm{i}y \bmod N) = y , \text{ for all } x, y \in N_{\mathbb{R}} .$$

If $\alpha \in M$, and $\mathfrak{X}^{\alpha} : T \longrightarrow \mathbb{C}^*$ is the corresponding function (as in TE I, it is useful to think of M as an additive group, and hence to adopt exponential notation for the characters regarded as functions on T), we obtain the formula

$$\mathfrak{X}^{\alpha}(x + \mathrm{i}y \bmod N) = \mathrm{e}^{2\pi\mathrm{i}(\langle\alpha,x\rangle + \mathrm{i}\langle\alpha,y\rangle)} , \text{ for all } x, y \in N_{\mathbb{R}} ;$$

hence

$$|\mathfrak{X}^{\alpha}(z)| = \mathrm{e}^{-2\pi\langle\alpha,\mathrm{ord}\, z\rangle} , \text{ for all } z \in T .$$

Next, in TE I, Ch. I, §1, we define embeddings of T in normal affine varieties X_{σ}, with the action of T on itself extending to an action of T on X_{σ}, whenever $\sigma \subset N_{\mathbb{R}}$ is a closed rational polyhedral cone not containing a line. Recall that

$$X_{\sigma} = \mathrm{Spec}\, \mathbb{C}[\ldots, \mathfrak{X}^{\alpha}, \ldots]_{\alpha \in M \cap \check{\sigma}} ;$$

here $\check{\sigma} \subset M_{\mathbb{R}}$ is the dual cone to σ, so $M \cap \check{\sigma}$ is a sub-semigroup of M. In order to study convergence in the classical topology and other details on X_{σ}, it will be convenient to introduce here the topological space (in the classical, not Zariski, topology) obtained by dividing X_{σ} by T_c. This will look like $N_{\mathbb{R}}$ with points at infinity added. Let us first construct these embeddings, which we call N_{σ}, of $N_{\mathbb{R}}$ and then show there is a map $\mathrm{ord} : X_{\sigma} \longrightarrow N_{\sigma}$ inducing a homeomorphism $X_{\sigma}/T_c \xrightarrow{\sim} N_{\sigma}$.

The simplest way to define N_σ is via a basis $\alpha_1, \ldots, \alpha_m$ of the semigroup $\check{\sigma} \cap M$. Then define

$$\begin{aligned} i : N_{\mathbb{R}} &\longrightarrow \mathbb{R}^m_{>0}, \\ x &\longmapsto (\mathrm{e}^{-2\pi\langle \alpha_1, x\rangle}, \ldots, \mathrm{e}^{-2\pi\langle \alpha_m, x\rangle}), \end{aligned}$$

and let

$$N_\sigma = \text{closure of } iN_{\mathbb{R}} \text{ in } \mathbb{R}^m_{\geq 0} .$$

It is very easy to see that this space is independent of the choice of basis (check that if you add to the α_i one more α, then N_σ does not change). If we let $N_{\mathbb{R}}$ act on $\mathbb{R}^m$ by

$$x \cdot (y_1, \ldots, y_m) = (\mathrm{e}^{-2\pi\langle \alpha_1, x\rangle} y_1, \ldots, \mathrm{e}^{-2\pi\langle \alpha_m, x\rangle} y_m) ,$$

then N_σ is the closure of the orbit of $(1, 1, \ldots, 1)$. In particular, $N_{\mathbb{R}}$ acts on N_σ, extending its action on itself by translation. Exactly as in the theory of torus embeddings (see TE I, Ch. I, §1, Theorem 2), we can decompose N_σ into $N_{\mathbb{R}}$-orbits; these will correspond bijectively to the faces of σ, and each one will contain a unique point $(y_1, \ldots, y_m)$ with $y_i = 0$ or 1 for all i. Explicitly, for every face τ of σ, the corresponding orbit is:

$$\begin{aligned} O(\tau) &= \left\{ (y_1, \ldots, y_m) \in N_\sigma \;\middle|\; \begin{matrix} y_i = 0 \text{ if } \alpha_i > 0 \text{ on } \mathrm{Int}\,\tau \\ y_i \neq 0 \text{ if } \alpha_i \equiv 0 \text{ on } \mathrm{Int}\,\tau \end{matrix} \right\} \\ &= N_{\mathbb{R}}\text{-orbit of } \varepsilon_\tau = (\varepsilon_1, \ldots, \varepsilon_m) , \end{aligned}$$

where

$$\varepsilon_i = \begin{cases} 0 & \text{if } \alpha_i > 0 \text{ on } \mathrm{Int}\,\tau \\ 1 & \text{if } \alpha_i \equiv 0 \text{ on } \mathrm{Int}\,\tau . \end{cases}$$

This can be proven following TE I, substituting the following lemma for the use of $k[[t]]$.

Lemma 1.1 *If $\{x_k\}$ is a sequence in $N_{\mathbb{R}}$ and $S \subset \{1, \ldots, m\}$ satisfies*

$$\lim_{k \longrightarrow \infty} \alpha_i(x_k) = \lambda_i \ , \ i \in S,$$
$$\lim_{k \longrightarrow \infty} \alpha_i(x_k) = \infty \ , \ i \notin S \ ,$$

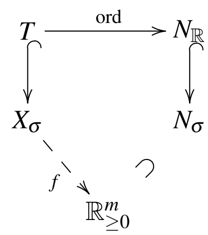
then

(a) *there is some $y \in N_{\mathbb{R}}$ with $\alpha_i(y) = 0$ for $i \in S$; $\alpha_i(y) > 0$ for $i \notin S$;*

(b) *there is some $z \in N_{\mathbb{R}}$ with $\alpha_i(z) = \lambda_i$ for $i \in S$.*

Proof Left to reader. □

Now if we map X_σ into $\mathbb{R}^m_{\geq 0}$ as follows:



$$f(x) = (|\mathfrak{X}^{\alpha_1}(x)|, \dots, |\mathfrak{X}^{\alpha_m}(x)|) ,$$

we get a commutative diagram. Since T is dense in X_σ, it follows that f defines a map

$$\text{ord} : X_\sigma \longrightarrow N_\sigma$$

and that $\text{ord}(gx) = \text{ord}(x)$ for all $g \in T_c$. Conversely, if $\text{ord}(x_1) = \text{ord}(x_2)$, it follows that $|\mathfrak{X}^\alpha(x_1)| = |\mathfrak{X}^\alpha(x_2)|$ for all $\alpha \in \check{\sigma} \cap M$, from which it follows readily that $x_1 = gx_2$ for some $g \in T_c$. Note that if $\mathbb{O}^\tau \subset X_\sigma$ is the orbit corresponding to τ, then $\text{ord}^{-1}(O(\tau)) = \mathbb{O}^\tau$.

For some purposes, it is convenient to have a coordinate-invariant way of describing N_σ as $N_\mathbb{R}$ plus a set of ideal points at infinity. To describe N_σ this way, for every face τ of σ, let

$$L(\tau) = \text{smallest linear space containing } \tau .$$

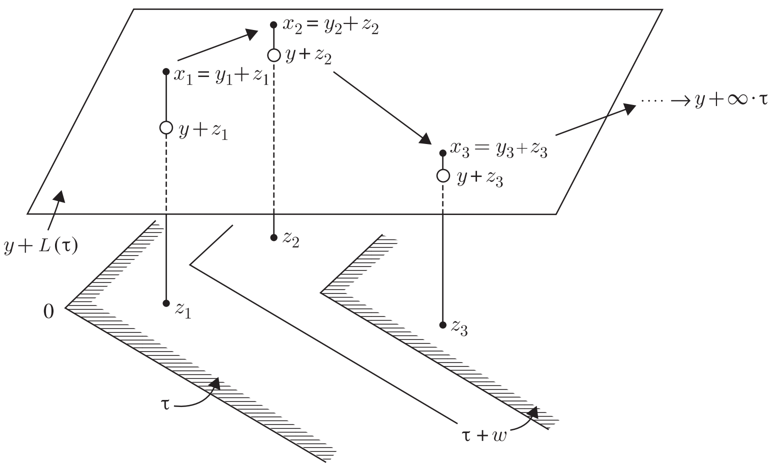
Then $L(\tau)$ is the stabilizer of ε_τ, so we get:

$$\begin{aligned} N_\mathbb{R}/L(\tau) &\xrightarrow{\sim} O(\tau) \\ x &\longmapsto x \cdot \varepsilon_\tau . \end{aligned}$$

Let $x + \infty \cdot \tau \in N_\sigma$ denote $x \cdot \varepsilon_\tau$ (where $x_1 + \infty \cdot \tau = x_2 + \infty \cdot \tau$ if and only if $x_1 - x_2 \in L(\tau)$). The reason for this notation is as follows: decompose $N_\mathbb{R} = N'_\mathbb{R} \oplus L(\tau)$, choose any sequence $x_n = y_n + z_n \in N_\mathbb{R} = N'_\mathbb{R} \oplus L(\tau)$, and choose any $y \in N'_\mathbb{R}$. Then one sees easily that

$$\left[\lim_{n\to\infty} x_n = y + \infty \cdot \tau \text{ in } N_\sigma\right] \Longleftrightarrow \left[\begin{array}{c} \lim_{n\to\infty} y_n = y \text{ and, for every} \\ w \in L(\tau),\, z_n \in \tau + w \text{ if } n \gg 0 . \end{array}\right]$$

Heuristically, we have added a lower-dimensional vector space isomorphic to $N_\mathbb{R}/L(\tau)$ of ideal points $x + \infty \cdot \tau$ obtained by starting at x and moving out to infinity in the direction determined by the cone τ.



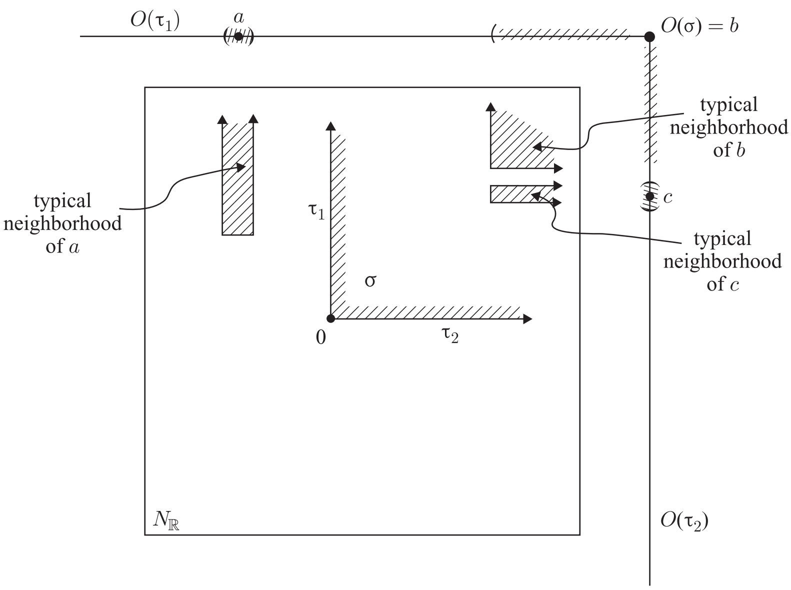
Our convergence condition may be rephrased by saying that a fundamental system of neighborhoods of $y + \infty \cdot \tau$ in $N_{\mathbb{R}}$ is given by

$$U^0_{\varepsilon,w}(y + \infty \cdot \tau) = y + w + B_\varepsilon + \tau ,$$

for any $w \in L(\tau)$ and any $\varepsilon > 0$, where B_ε denotes the ε-ball around 0 (take any metric on $N_{\mathbb{R}}$). More generally, with this notation, a fundamental system of neighborhoods of $y + \infty \cdot \tau$ in N_σ is given by

$$U_{\varepsilon,w}(y + \infty \cdot \tau) = U^0_{\varepsilon,w}(y + \infty \cdot \tau) \cup \bigcup_{\tau' \text{ face of } \tau} (y + w + B_\varepsilon + \tau + \infty \cdot \tau') .$$

For instance, if $N_{\mathbb{R}} = \mathbb{R}^2$ and σ is the positive quadrant, we get the following picture:



Next recall that in TE I, Ch. I, §2, we glue the affine varieties X_σ together: whenever $\{\sigma_\alpha\}$ is a *rational partial polyhedral decomposition of* $N_{\mathbb{R}}$, meaning

(i) if σ is a face of σ_α, then $\sigma = \sigma_\beta$, for some β;
(ii) for all α, β, the cone $\sigma_\alpha \cap \sigma_\beta$ is a face of σ_α and σ_β,

then we can glue the X_{σ_α} together, obtaining a scheme $X_{\{\sigma_\alpha\}}$. In TE I, we asked that $\{\sigma_\alpha\}$ be a finite set, so that $X_{\{\sigma_\alpha\}}$ was a variety. This is in fact totally irrelevant: for any set $\{\sigma_\alpha\}$ as above, we get an $X_{\{\sigma_\alpha\}}$ as before, except that it may require an infinite number of affines to cover it. Now $X_{\{\sigma_\alpha\}}$ is always a separated normal irreducible scheme, *locally* of finite type over $\mathbb{C}$ and containing T as an open dense subset. In exactly the same way, we glue the N_{σ_α} together into a topological space $N_{\{\sigma_\alpha\}}$, which is $N_{\mathbb{R}}$ plus a large number of ideal vector spaces situated at infinity in many different directions. Moreover, we glue the ord maps together into one map:

$$\text{ord} : X_{\{\sigma_\alpha\}} \longrightarrow N_{\{\sigma_\alpha\}} .$$

For instance, $X_{\{\sigma_\alpha\}}$, as a set, is the disjoint union of T-orbits $\mathbb{O}^{\sigma_\alpha}$, one for each α; likewise $N_{\{\sigma_\alpha\}}$ as a set is the disjoint union of $N_{\mathbb{R}}$-orbits $O(\sigma_\alpha)$, one for each α, and $\text{ord}^{-1}(O(\sigma_\alpha)) = \mathbb{O}^{\sigma_\alpha}$.

2 The functor of a torus embedding

In order to make some of our later constructions of compactifications D/Γ purely algebraic and valid for schemes over any ground fields, it will be useful to learn what functor a torus embedding represents. This also gives us another view of what torus embeddings are. First some notations and definitions.

(1) If S is a scheme and X is a set, X_S denotes the constant sheaf on S with stalk X.
(2) Every semigroup or sheaf of semigroups will have an identity element e or identity section e.
(3) If A_1, A_2 are semigroups, a homomorphism $\phi : A_1 \longrightarrow A_2$ is called *strict* if $\phi(e_1) = e_2$ and $\phi(x)$ invertible implies x invertible. If A_1, A_2 are sheaves of semigroups on S, we require that, for every $s \in S$, the map on stalks $\phi_s : A_{1,s} \longrightarrow A_{2,s}$ is strict.
(4) If S is a scheme, then $\mathscr{O}_S^{(\times)}$ will be the semigroup sheaf $(\mathscr{O}_S, \text{mult.})$.

The result is:

Theorem 2.1 *Let T be a torus over k and $T \subset X_{\{\sigma_\alpha\}}$ a torus embedding, where $\sigma_\alpha \subset N(T)_{\mathbb{R}}$ are polyhedral cones. For any k-scheme S, let $F_{\{\sigma_\alpha\}}(S)$ be the set of pairs (Σ, π) consisting of a sub-semigroup sheaf $\Sigma \subset M(T)_S$ and a strict homomorphism $\pi : \Sigma \longrightarrow \mathscr{O}_S^{(\times)}$ such that, for all $s \in S$, we have $\Sigma_s = \check{\sigma}_\alpha \cap M(T)$ for some α. Then there are canonical isomorphisms, functorial in k-schemes S:*

$$\operatorname{Hom}_k(S, X_{\{\sigma_\alpha\}}) \cong F_{\{\sigma_\alpha\}}(S) .$$

Proof We first show how to associate a pair (Σ, π) to a morphism $f : S \longrightarrow X_{\{\sigma_\alpha\}}$. Define:

$$U_\alpha = f^{-1}(X_{\sigma_\alpha}) ,$$
$$\Sigma = \text{the union of the subsheaves } (\check{\sigma}_\alpha \cap M(T))_{U_\alpha} \text{ of } M(T)_S .$$

Note that, for all $s \in S$, if $f(s) \in \mathbb{O}^\alpha$, then

$$\begin{aligned} s \in U_\beta &\iff f(s) \in X_{\sigma_\beta} \\ &\iff \mathbb{O}^\alpha \subset X_{\sigma_\beta} \\ &\iff \sigma_\alpha \text{ is a face of } \sigma_\beta \\ &\iff \check{\sigma}_\beta \cap M(T) \subseteq \check{\sigma}_\alpha \cap M(T) ; \end{aligned}$$

hence the stalk of Σ at s is the union of the subsets $\check{\sigma}_\beta \cap M(T)$ of $M(T)$ for all σ_β with face σ_α, i.e., just $\check{\sigma}_\alpha \cap M(T)$. Hence if $r \in \Sigma_s$, then $r \in \check{\sigma}_\alpha$, so

$\mathfrak{X}^r$ is defined on X_{σ_α} and $f^*(\mathfrak{X}^r)$ is defined at s. Therefore we can define $\pi : \Sigma \longrightarrow \mathscr{O}_S^{(\times)}$ by

$$\pi(r) = f^*(\mathfrak{X}^r) \,.$$

Note that

$$\begin{aligned} \pi(r) \text{ invertible in } \mathscr{O}_{S,s} &\iff \pi(r)(s) \neq 0 \\ &\iff \mathfrak{X}^r(f(s)) \neq 0 \\ &\iff \mathfrak{X}^r \not\equiv 0 \text{ on } \mathbb{O}^\alpha \\ &\iff r \equiv 0 \text{ on } \sigma_\alpha \\ &\iff -r \in \check{\sigma}_\alpha \cap M(T) \\ &\iff r \text{ invertible in } \Sigma_s \,, \end{aligned}$$

hence π is a strict homomorphism.

Next, let us start with (Σ, π) and define a morphism f. Define open sets U_α by

$$U_\alpha = \{ s \in S \mid \check{\sigma}_\alpha \cap M(T) \subset \Sigma_s \} \,.$$

These form an open covering of S such that if σ_α is a face of σ_β, then $U_\beta \subset U_\alpha$. Next define

$$f_\alpha : U_\alpha \longrightarrow X_{\sigma_\alpha} = \operatorname{Spec} k[\ldots, \mathfrak{X}^r, \ldots]_{r \in \check{\sigma}_\alpha \cap M(T)}$$

via $f_\alpha^*(\mathfrak{X}^r) = \pi(r)$ for all $r \in \check{\sigma}_\alpha \cap M(T)$: this is correct since such an r is in $\Gamma(U_\alpha, \Sigma)$ and since $\pi(r_1 + r_2) = \pi(r_1) \cdot \pi(r_2)$. Now, for any α and β, let $\sigma_\gamma = \sigma_\alpha \cap \sigma_\beta$, which is a face of σ_α and σ_β. Then

$$U_\alpha \cap U_\beta = \{ s \in S \mid \check{\sigma}_\alpha \cap M(T) \subset \Sigma_s \text{ and } \check{\sigma}_\beta \cap M(T) \subset \Sigma_s \} \,.$$

But if $\Sigma_s = \check{\sigma}_\delta \cap M(T)$, then

$$\begin{aligned} \begin{bmatrix} \Sigma_s \supset \check{\sigma}_\alpha \cap M(T) \text{ and} \\ \Sigma_s \supset \check{\sigma}_\beta \cap M(T) \end{bmatrix} &\iff \check{\sigma}_\delta \supset \check{\sigma}_\alpha \text{ and } \check{\sigma}_\delta \supset \check{\sigma}_\beta \\ &\iff \sigma_\delta \subset \sigma_\alpha \text{ and } \sigma_\delta \subset \sigma_\beta \\ &\iff \sigma_\delta \subset \sigma_\gamma \\ &\iff \Sigma_s \subset \check{\sigma}_\gamma \cap M(T) \,, \end{aligned}$$

so $U_\alpha \cap U_\beta = U_\gamma$. Finally, it is clear from the definition that $f_\alpha = \operatorname{res} f_\beta$ whenever $U_\alpha \subset U_\beta$. Therefore the f_α patch together to form a morphism $f : S \longrightarrow X_{\{\sigma_\alpha\}}$.

It is now straightforward to check that these two procedures – associating a (Σ, π) to an f and associating an f to a (Σ, π) – are inverse to each other: we leave this to the reader. □

For instance, we find:

$$X_{\{\sigma_\alpha\}}(k) \cong \{(\alpha,\pi) \mid \pi : \check{\sigma}_\alpha \cap M(T) \longrightarrow k^{(\times)} \text{ strict homomorphism}\} .$$

If $k = \mathbb{C}$, one can easily prove also that

$$\begin{aligned} N_{\{\sigma_\alpha\}} &\cong \{(\alpha,\rho) \mid \rho : \check{\sigma}_\alpha \cap M(T) \longrightarrow \mathbb{R}^{(\times)}_{\geq 0} \text{ strict homomorphism}\} \\ &\cong \{(\alpha,\sigma) \mid \sigma : \check{\sigma}_\alpha \cap M(T) \longrightarrow \mathbb{R} \cup \{\infty\} \text{ strict homomorphism}\} , \end{aligned}$$

where $\mathbb{R} \cup \{\infty\}$ is a semigroup via $+$. Here

$$\mathrm{ord} : X_{\{\sigma_\alpha\}} \longrightarrow N_{\{\sigma_\alpha\}}$$

is given by

$$\begin{aligned} \rho(x) &= |\pi(x)| , \\ \sigma(x) &= -\log \rho(x) . \end{aligned}$$

3 Toroidal embeddings over the complex numbers

We wish to review here quickly some results of TE I, Ch. II, indicating ways to interpret them over $\mathbb{C}$, and generalizing them slightly. A pair

$$U \subset X,$$

where U is a Zariski-open subset of a normal variety X, was called a *toroidal embedding* if, for all $x \in X$, we have that (X,U) is formally isomorphic at x to (X_σ, T) at some $t \in X_\sigma$ (for some torus embedding $T \subset X_\sigma$). Equivalently, this means that there is an étale correspondence between X and X_σ, relating x and t, with U and T corresponding open sets. Over $\mathbb{C}$, a pair

$$U \subset X ,$$

where X is an analytic space and U is open in the complex topology, will be called a *toroidal embedding* if, for all $x \in X$, there exists a small neighborhood $W_x \subset X$ of x such that $(W_x, W_x \cap U)$ is isomorphic to $(V_t, V_t \cap T)$ for some neighborhood $V_t \subset X_\sigma$ of some $t \in X_\sigma$ (for some torus embedding $T \subset X_\sigma$). When X, U are varieties, this coincides with the previous definition. Now, this implies immediately that W_x has a canonical stratification $\{Y_{\alpha,x}\}$ into non-singular locally closed analytic strata with $\overline{Y}_{\alpha,x}$ normal: let E_i be the irreducible components of $W_x \setminus W_x \cap U$, and let the $Y_{\alpha,x}$ be the sets

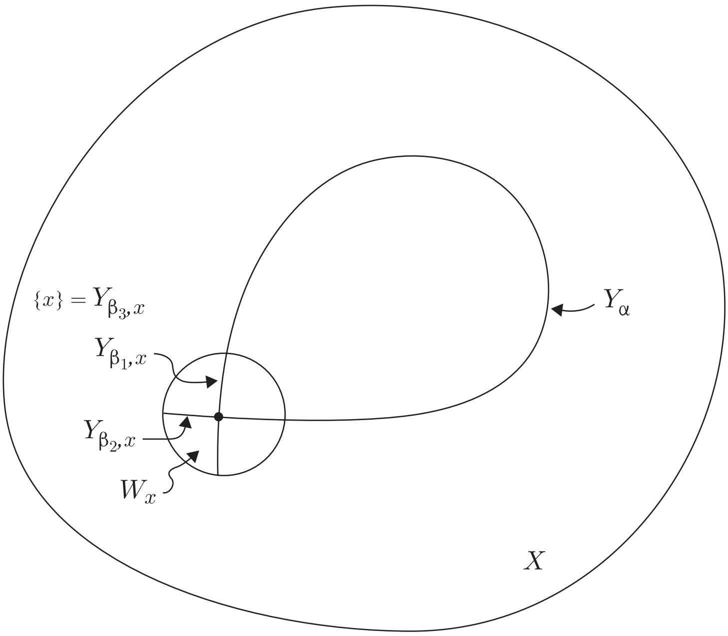
$$\bigcap_{i \in I} E_i \setminus \bigcup_{i \notin I} E_i .$$

We shrink W_x if necessary, so that these $Y_{\alpha,x}$ are connected. As x varies, these strata patch up on overlaps, so we can uniquely stratify the whole of X into

$\{Y_\alpha\}$, where the Y_α are connected, locally closed, non-singular analytic strata, and where $Y_\alpha \cap W_x$ is a union of the $Y_{\beta,x}$. However, it may happen that

$$Y_\alpha \cap W_x \supset \text{more than one } Y_{\beta,x} \,.$$

This means that there is a path in X starting and ending in W_x and lying all in one stratum, but linking two distinct local strata:



Since this will mean that $\overline{Y}_\alpha$ has more than one branch through x, it is equivalent to $\overline{Y}_\alpha$ being non-normal. As in TE I, p. 57, we say that (X,U) has or has not *self-intersection* according to whether $Y_\alpha \cap W_x$ can be more than one local stratum, or $Y_\alpha \cap W_x$ is always one local stratum. In TE I, we stuck with (X,U)'s without self-intersection. However, there is a class of toroidal embeddings with self-intersection that are almost as nice and that arise in the examples we will treat. Suppose $Y_{\beta_1,x}$ and $Y_{\beta_2,x}$ are part of the same global stratum Y_α. Locally at x there is a unique stratum $Y_{\beta_3,x}$ such that

$$\overline{Y}_{\beta_3,x} = \overline{Y}_{\beta_1,x} \cap \overline{Y}_{\beta_2,x} \,.$$

Let $Y_{\beta_3,x}$ define a global stratum Y_γ. We say that (X,U) is *without monodromy* if Y_γ has a neighborhood W such that $Y_{\beta_1,x}$ and $Y_{\beta_2,x}$ lie in different components of $Y_\alpha \cap W$. To visualize this, note that, for every path in Y_γ beginning and ending at x, we can uniquely propagate the germ of analytic space $\overline{Y}_{\beta_1,x}$ along this path. If this germ can be taken to $\overline{Y}_{\beta_2,x}$ by such a path, then, for every