Uniform Central Limit Theorems

Second Edition

This work about probability limit theorems for empirical processes on general spaces, by one of the founders of the field, has been considerably expanded and revised from the original edition. When samples become large, laws of large numbers and central limit theorems are guaranteed to hold uniformly over wide domains. The author gives a thorough treatment of the subject, including an extended treatment of Vapnik–Červonenkis combinatorics, the Ossiander L2 bracketing central limit theorem, the Giné–Zinn bootstrap central limit theorem in probability, the Bronstein theorem on approximation of convex sets, and the Shor theorem on rates of convergence over lower layers. This new edition contains proofs of several main theorems not proved in the first edition, including the Bretagnolle–Massart theorem giving constants in the Komlós–Major–Tusnády rate of convergence for the classical empirical process, Massart's form of the Dvoretzky–Kiefer–Wolfowitz inequality with precise constant, Talagrand's generic chaining approach to boundedness of Gaussian processes, a characterization of uniform Glivenko–Cantelli classes of functions, Giné and Zinn's characterization of uniform Donsker classes of functions (i.e., classes for which the central limit theorem holds uniformly over all probability measures P), and the Bousquet–Koltchinskii–Panchenko theorem that the convex hull of a uniform Donsker class is uniform Donsker.

The book will be an essential reference for mathematicians working in infinite-dimensional central limit theorems, mathematical statisticians, and computer scientists working in computer learning theory. Problems are included at the end of each chapter so the book can also be used as an advanced text.

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Massachusetts Institute of Technology

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Preface to the Second Edition

This book developed out of some topics courses given at M.I.T. and my lectures at the St.-Flour probability summer school in 1982. The material of the book has been expanded and extended considerably since then. At the end of each chapter are some problems and notes on that chapter.

Starred sections are not cited later in the book except perhaps in other starred sections. The first edition had several double-starred sections in which facts were stated without proofs. This edition has no such sections.

The following, not proved in the first edition, now are: (i) for Donsker's theorem on the classical empirical process $\alpha_n := \sqrt{n}(F_n - F)$, and the Komlós–Major–Tusnády strengthening to give a rate of convergence, the Bretagnolle–Massart proof with specified constants; (ii) Massart's form of the Dvoretzky–Kiefer–Wolfowitz inequality for α_n with optimal constant; (iii) Talagrand's generic chaining approach to boundedness of Gaussian processes, which replaces the previous treatment of majorizing measures; (iv) characterization of uniform Glivenko–Cantelli classes of functions (from a paper by Dudley, Giné, and Zinn, but here with a self-contained proof); (v) Giné and Zinn's characterization of uniform Donsker classes of functions; (vi) its consequence that uniformly bounded, suitably measurable classes of functions satisfying Pollard's entropy condition are uniformly Donsker; and (vii) Bousquet, Koltchinskii, and Panchenko's theorem that a convex hull preserves the uniform Donsker property.

The first edition contained a chapter on invariance principles, based on a 1983 paper with the late Walter Philipp. Some techniques introduced in that paper, such as measurable cover functions, are still used in this book. But I have not worked on invariance principles as such since 1983. Much of the work on them treats dependent random variables, as did parts of the 1983 paper which Philipp contributed. The present book is mainly about the i.i.d. case. So I suppose the chapter is outdated, and I omit it from this edition.

For useful conversations and suggestions on topics in the book I'm glad to thank Kenneth Alexander, Niels Trolle Andersen, the late Miguel Arcones, Patrice Assouad, Erich Berger, Lucien Birgé, Igor S. Borisov, Donald Cohn, Yves Derrienic, Uwe Einmahl, Joseph Fu, Sam Gutmann, David Haussler, Jørgen Hoffmann-Jørgensen, Yen-Chin Huang, Vladimir Koltchinskii, the late Lucien Le Cam, David Mason, Pascal Massart, James Munkres, Rimas Norvaiša, the late Walter Philipp, Tom Salisbury, the late Rae Shortt, Michel Talagrand, Jon Wellner, He Sheng Wu, Joe Yukich, and Joel Zinn. I especially thank Denis Chetverikov, Peter Gaenssler and Franz Strobl, Evarist Giné, and Jinghua Qian, for providing multiple corrections and suggestions. I also thank Xavier Fernique (for the first edition), Evarist Giné (for both editions), and Xia Hua (for the second edition) for giving or sending me copies of expositions.

Notes

Throughout this book, all references to "RAP" are to the author's book *Real Analysis and Probability*, second edition, Cambridge University Press, 2002.

Also, "$A := B$" means A is defined by B, whereas "$A =: B$" means B is defined by A.