Combinatorics: The Rota Way

Gian-Carlo Rota was one of the most original and colorful mathematicians of the twentieth century. His work on the foundations of combinatorics focused on revealing the algebraic structures that lie behind diverse combinatorial areas and created a new area of algebraic combinatorics. His graduate courses influenced generations of students.

Written by two of his former students, this book is based on notes from his courses and on personal discussions with him. Topics include sets and valuations, partially ordered sets, distributive lattices, partitions and entropy, matching theory, free matrices, doubly stochastic matrices, Möbius functions, chains and antichains, Sperner theory, commuting equivalence relations and linear lattices, modular and geometric lattices, valuation rings, generating functions, umbral calculus, symmetric functions, Baxter algebras, unimodality of sequences, and location of zeros of polynomials. Many exercises and research problems are included and unexplored areas of possible research are discussed.

This book should be on the shelf of all students and researchers in combinatorics and related areas.

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GIAN-CARLO ROTA (1932–1999) was a professor of applied mathematics and natural philosophy at the Massachusetts Institute of Technology. He was a member of the National Academy of Science. He was awarded the 1988 Steele Prize of the American Mathematical Society for his 1964 paper "On the Foundations of Combinatorial Theory I. Theory of Möbius Functions." He was a founding editor of *Journal of Combinatorial Theory*.

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Gian-Carlo Rota, Circa 1970
Pencil drawing by Eleanor Blair

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Frontmatter
More information

Contents

Preface

The working title of this book was "Combinatorics 18.315." In the private language of the Massachusetts Institute of Technology, Course 18 is Mathematics, and 18.315 is the beginning graduate course in combinatorial theory. From the 1960s to the 1990s, 18.315 was taught primarily by the three permanent faculty in combinatorics, Gian-Carlo Rota, Daniel Kleitman, and Richard Stanley. Kleitman is a problem solver, with a prior career as a theoretical physicist. His way of teaching 18.315 was intuitive and humorous. With Kleitman, mathematics is fun. The experience of a Kleitman lecture can be gleaned from the transcripts of two talks.[1] Stanley's way is the opposite of Kleitman. His lectures are careful, methodical, and packed with information. He does not waste words. The experience of a Stanley lecture is captured in the two books *Enumerative Combinatorics I* and *II*, now universally known as *EC1* and *EC2*. Stanley's work is a major factor in making algebraic combinatorics a respectable flourishing mainstream area.

It is difficult to convey the experience of a Rota lecture. Rota once said that the secret to successful teaching is to reveal the material so that at the end, the idea – and there should be only one per lecture – is obvious, ready for the audience to "take home." We must confess that we have failed to pull this off in this book. The immediacy of a lecture cannot (and should not) be frozen in the textuality of a book. Instead, we have tried to convey the method behind Rota's research. Although he would object to it being stated in such stark simplistic terms, mathematical research is not about *solving* problems; it is about *finding* the right problems. One way of finding the right problems is to look for ideas common to subjects, ranging from, say, category theory to statistics. What is shared may be the implicit algebraic structures that hide behind the technicalities, in which case finding the structure is part

[1] Kleitman (1979, 2000).

of "applied universal algebra." The famous paper *Foundations I*, which revealed the role of partially ordered sets in combinatorics, is a product of this point of view. To convey Rota's thinking, which involves all of mathematics, one must go against an *idée reçue* of textbook writing: the prerequisites for this book are, in a sense, all mathematics. However, it is the ideas, not the technical details, that matter. Thus, in a different sense, there are no prerequisites to this book: we intend that a minimum of technical knowledge is needed to seriously appreciate the text of this book. Those parts where special technical knowledge is needed, usually in the exercises, can be skimmed over.

Rota taught his courses with different topics and for different audiences. The chapters in this book reflect this. Chapter 1 is about sets, functions, relations, valuations, and entropy. Chapter 2 is mostly a survey of matching theory. It provides a case study of Rota's advice to read on the history of a subject before tackling its problems. The aim of Chapter 2 is to find what results one should expect when one extends matching theory to higher dimensions. Possible paths are suggested in Section 2.8. The third chapter offers a mixture of topics in partially ordered sets. The first section is about Möbius functions. After the mid-1960s, Möbius functions were never the focus of a Rota course; his feeling was that he had made his contribution. However, a book on Rota's combinatorics would be incomplete without Möbius functions. Other topics in Chapter 3 are Dilworth's chain partition theorem; Sperner theory; modular, linear, and geometric lattices; and valuation rings. Linear lattices, or lattices represented by commuting equivalence relations, lie at the intersection of geometric invariant theory and the foundations of probability theory. Chapter 4 is about generating functions, polynomial sequences of binomial type, and the umbral calculus. These subjects have been intensively studied and the chapter merely opens the door to this area. Chapter 5 is about symmetric functions. We define them by distribution and occupancy and apply them to the study of Baxter algebras. This chapter ends with a section on symmetric functions over finite fields. The sixth chapter is on polynomials and their zeros. The topic is motivated, in part, by unimodality conjectures in combinatorics and was the last topic Rota taught regularly. Sadly, we did not have the opportunity to discuss this topic in detail with him.

There is a comprehensive bibliography. Items in the bibliography are referenced in the text by author and year of publication. In a few cases when two items by the same authors are published in the same year, suffixes *a* and *b* are appended according to the order in which the items are listed. Exceptions are several papers by Rota and the two volumes of his selected papers; these are referenced by short titles. Our convention is explained in the beginning of the bibliography.

We should now explain the authorship and the title of this book. Gian-Carlo Rota passed away unexpectedly in 1999, a week before his 67th birthday. This book was physically written by the two authors signing this preface. We will refer to the third author simply as Rota. As for the title, we wanted one that is not boring. The word "way" is not meant to be prescriptive, in the sense of "my way or the highway." Rather, it comes from the core of the cultures of the three authors. The word "way" resonates with the word "cammin" in the first line of Dante's *Divina commedia*, "Nel mezzo del cammin di nostra vita." It also resonates as the character "tao" in Chinese. In both senses, the way has to be struggled for and sought individually. This is best expressed in Chinese:

道可道，非常道

Inadequately translated into rectilinear English, this says "a way which can be *wayed* (that is, taught or followed) cannot be a way." Rota's way is but one way of doing combinatorics. After "seeing through" Rota's way, the reader will seek his or her own way.

It is our duty and pleasure to thank the many friends who have contributed, knowingly or unknowingly, to the writing of this book. There are several sets of notes from Rota's courses. We have specifically made use of our own notes (1976, 1977, 1994, and 1995), and more crucially, our recollection of many conversations we had with Rota. Norton Starr provided us with his notes from 1964. These offer a useful pre-foundations perspective. We have also consulted notes by Miklós Bona, Gabor Hetyei, Richard Ehrenborg, Matteo Mainetti, Brian Taylor, and Lizhao Zhang from the early 1990s. We have benefited from discussions with Ottavio D'Antona, Wendy Chan, and Dan Klain. John Guidi generously provided us with his verbatim transcript from 1998, the last time Rota taught 18.315. Section 1.4 is based partly on notes of Kenneth Baclawski, Sara Billey, Graham Sterling, and Carlo Mereghetti. Section 2.8 originated in discussions with Jay Sulzberger in the 1970s. Sections 3.4 and 3.5 were much improved by a discussion with J. B. Nation. William Y. C. Chen and his students at the Center for Combinatorics at Nankai University (Tianjin, China) – Thomas Britz, Dimitrije Kostic, Svetlana Poznanovik, and Susan Y. Wu – carefully read various sections of this book and saved us from innumerable errors. We also thank Ester Rota Gasperoni, Gian-Carlo's sister, for her encouragement of this project.

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