1
Introduction

1.1 Introduction

The main purpose of writing this book is to convey to the general mathematical audience the notion of a Zariski geometry with the whole spectrum of geometric ideas arising in model-theoretic context. The idea of a Zariski geometry is intrinsically linked with algebraic geometry, as are many other model-theoretic geometric ideas. However, there are also very strong links with combinatorial geometries, such as matroids (pre-geometries) and abstract incidence systems. Model theory developed a very general unifying point of view based on the model-theoretic geometric analysis of mathematical structures as diverse as compact complex manifolds and general algebraic varieties, differential fields, difference fields, algebraic groups, and others. In all of these, Zariski geometries have been detected and have proved crucial for the corresponding theory and applications. In more recent works, this author has established a robust connection to non-commutative algebraic geometry.

Model theory has always been interested in studying the relationship between a mathematical structure, such as the field of complex numbers $(\mathbb{C}, +, \cdot)$, and its description in a formal language, such as the finitary language suggested by D. Hilbert: *the first-order language*. The best possible relationship is when a structure M is the unique, up-to-isomorphism model of the description $\mathrm{Th}(M)$: *the theory of* M. Unfortunately, for a first-order language, this is the case only when M is finite because in the first order, it is impossible to fix an infinite cardinality of (the universe of) M. So, the next best relationship is when the isomorphism type of M is determined by $\mathrm{Th}(M)$ and the cardinality λ of M (λ-categoricity), such as $\mathrm{Th}(\mathbb{C}, +, \cdot)$ – the theory of the field of complex numbers, in which 'complex algebraic geometry lives'. Especially interesting is the case when λ is uncountable and the description is at most countable. In fact, in this case, Morley's theorem (1965) states that the theory Th(M) is not sensitive to a particular choice of λ; it has a unique model in every uncountable cardinality.

The proof of Morley's theorem marked the beginning of stability theory, which studies categorical theories in uncountable cardinals and generalisations (*every categorical theory in uncountable cardinals is stable*). Categoricity and stability turned out to be amazingly effective classification principles. To sum up the results of 40 years of research in a few lines, we lay out the following conclusions:

1. There is a clear hierarchy of the 'logical perfection' of a theory in terms of stability. Categorical theories and their models are at the top of this hierarchy.
2. The key feature of stability theory is dimension theory and, linked to it, dependence theory resembling the dimension theory of algebraic geometry and the theory of algebraic dependence in fields. In fact, algebraic geometry and its related areas are the main sources of examples.
3. There has been considerable progress toward the classification of structures with stable and, especially, uncountably categorical theories. The (fine) classification theory makes use of certain geometric principles, both classical and those specifically developed in model theory. These geometric principles proved useful in applications, such as in Diophantine geometry.

In classical mathematics, three basic types of dependencies are known:

1. algebraic dependence in the theory of fields;
2. linear dependence in the theory of vector spaces; and
3. dependence of trivial (combinatorial) type (e.g. two vertices of a graph are dependent if they belong to the same connected component).

One of the useful conjectures in fine classification theory was the **trichotomy principle**, which states that every dependence in an uncountably categorical theory is based on one of three classical types.

A more elaborate form of this conjecture implies that any uncountably categorical structure with a non-linear, non-trivial geometry comes from algebraic geometry over an algebraically closed field. (It makes sense to call a dependence type *non-linear* if it does not belong to types 2 and 3.) For example, a special case of this conjecture has been known since 1975 and is still open (see survey by Cherlin, 2002).

The algebraicity conjecture: Suppose $(G, \cdot)$ is a simple group with Th(G) categorical in uncountable cardinals. Then $G = \mathrm{G}(K)$ for some simple algebraic group G and an algebraically closed field K.

The trichotomy principle proved to be false in general (Hrushovski, 1988) but nevertheless holds for many important classes. The notion of a Zariski structure was designed primarily to identify all such classes.

Originally, the idea of a Zariski structure was a condition which would isolate the 'best' possible classes on the top of the hierarchy of stable structures. Because it has been realised that purely logical conditions are not sufficient for the trichotomy principle to hold, it has also been realised that a topological ingredient added to the definition of a categorical theory might suffice. In fact, a very coarse topology similar to the Zariski topology in algebraic geometry is sufficient. Along with the introduction of the topology, one also postulates certain properties of it, mainly of how the topology interacts with the dimension notion. One of the crucial properties of this kind is in fact a weak form of smoothness of the geometry in question; in this book, it is called *the pre-smoothness property*.

In more detail, a (Noetherian) Zariski structure is a structure $\mathrm{M} = (M, \mathcal{C})$, on the universe M in the language given by the family of relations listed in $\mathcal{C}$.

For each n, the subsets of M^n corresponding to relations from $\mathcal{C}$ form a Noetherian topology. The topology is endowed with a dimension notion (e.g. the Krull dimension). Dimension is well behaved with respect to projections $M^{n+1} \to M^n$.

The structure M is said to be pre-smooth if for any two closed irreducible $S_1, S_2 \subseteq M^n$, and for any irreducible component S_0 of the set $S_1 \cap S_2$,

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim M^n.$$

It has been said already that the basic examples of pre-smooth Noetherian Zariski structures come from algebraic geometry. Indeed, let $M = M(K)$ be the set of K-points of a smooth algebraic variety over an algebraically closed field K. For $\mathcal{C}$, take the family of Zariski closed subsets (relations) of M^n, all n. Set $\dim S$ to be the Krull dimension. This is a pre-smooth Zariski structure (geometry).

Another important class of examples is the class of compact complex manifolds. Here M should be taken to be the underlying set of a manifold, and $\mathcal{C}$ should be taken as the family of all analytic subsets of M^n, all n.

Proper analytic varieties in the sense of *rigid analytic geometry* (analogues of compact complex manifolds for non-Archimedean valued fields) constitute yet another class of Noetherian Zariski structures.

It follows from the general theory developed in these lectures that all these structures (and Zariski structures in general) are at the top of the logical hierarchy (i.e. they have finite Morley rank and in most important cases are uncountably categorical). Interestingly, for the second and third classes, this is hard to establish without first checking that the structures are Zariski.

So far, the main result of the general theory is the classification of one-dimensional, pre-smooth, Noetherian Zariski geometries M:

If M *is non-linear, then there is an algebraically closed field K, a quasi-projective algebraic curve $C_M = C_M(K)$, and a surjective map*

$$p : M \to C_M$$

of a finite degree (i.e. $p^{-1}(a) \leq d$ for each $a \in C_M$) such that for every closed $S \subseteq M^n$, the image $p(S)$ is Zariski closed in C_M^n (in the sense of algebraic geometry); if $\hat{S} \subseteq C_M^n$ is Zariski closed, then $p^{-1}(\hat{S})$ is a closed subset of M^n (in the sense of the Zariski structure M*).*

In other words, M is almost an algebraic curve. In fact, it is possible to specify some extra geometric conditions for M which imply M is an exact algebraic curve (see [HZ]).

The proof of the classification theorem proceeds as follows (Chapters 8 and 10):

First, for general Zariski structures, we develop an *infinitesimal analysis* that culminates with the introduction of *local multiplicities* of covers (maps) and intersections and the proof of *the implicit function theorem*.

Next, we focus on a specific configuration in a one-dimensional M given by the two-dimensional pre-smooth 'plane' M^2 and an n-dimensional ($n \geq 2$) pre-smooth family L of curves on M^2. We use the local multiplicities of intersections to define what it means to say that two curves are *tangent at a given point*. This is well defined in non-singular points of the curves, but in general we need a more subtle notion. This is a technically involved concept of a *branch of a curve at a point*. Once this is properly defined, we develop a theory of tangency for branches and prove, in particular, that tangency between branches is an equivalence relationship.

Next, we treat branches of curves on the plane M^2 as (graphs of) *local functions* from an infinitesimal neighbourhood of a point on M onto another infinitesimal neighbourhood. One can prove that the composition of such local functions is well behaved with respect to tangency. In particular, with respect to composition modulo tangency, local functions form a local group (pre-group, or a 'group-chunk' in the terminology of A. Weil). A generalisation of a known proof by Weil produces a pre-smooth *Zariski group*, more specifically an Abelian group J of dimension 1.

We now replace the initial one-dimensional M by the more suitable Zariski curve J and repeat the construction on the plane J^2. Again, we consider the composition of local functions on J modulo tangency. However, this time we take into account the existing group structure on J and find that our new group operation interacts with the existing one in a nice way. More specifically, the new group structure acts (locally) on the existing one by (local) endomorphisms. Using again the generalisation of Weil's pre-group theorem, we find *a field K with a Zariski structure on it*.

Notice that at this stage we do not know if the Zariski structure on K is the classical (algebraic) one. Obviously, it contains all algebraic Zariski closed relations, but we need to see that there are no extra ones in the Zariski topology. For this purpose, we undertake an analysis of projective spaces $\mathbf{P}^n(K)$. First, we prove that $\mathbf{P}^n(K)$ are *weakly complete* in our Zariski topology, which is the property analogous to the classical completeness in algebraic geometry. Then, expanding the intersection theory of the first sections, we manage to prove a *generalisation of the Bezout theorem*. This theorem is key in proving *the generalisation of the Chow theorem*: every Zariski closed subset of $\mathbf{P}^n(K)$ is algebraic. (Note that $\mathbf{P}^n(\mathbb{C})$ is a compact complex manifold and that every analytic subset of it is Zariski closed according to our definition.) This immediately implies that the structure on K is purely algebraic.

It follows from the construction of K in M that there is a non-constant Zariski-continuous map $f : M \to K$, with the domain of definition open in M. Such maps we call *Z-meromorphic functions*. Based on the generalisation of Chow's theorem, we prove that the inseparable closure of the field $K_Z(M)$ of Z-meromorphic functions is isomorphic to the field of rational functions of a smooth algebraic curve C_M. By the same construction, we find a Zariski-continuous map $p : M \to C_M$ which satisfies the required properties. This completes the proof of the classification theorem.

The classification theorem asserts that in the one-dimensional case, a non-linear Zariski geometry is *almost* an algebraic curve. This statement is true completely in algebraic geometry, compact complex manifolds, and proper rigid analytic varieties; in the last two, this is due to the Riemann existence theorem. However, in the general context of Noetherian Zariski geometries, the adverb 'almost' cannot be omitted. In Section 5.1, we present a construction that provides examples of non-classical Noetherian Zariski geometries, that is, ones which are *not definable in an algebraically closed field*. We study a special but typical example and look for a way to explain the geometry of M in terms of co-ordinate functions to K and co-ordinate rings. We conclude that there are just not enough of regular (definable Zariski-continuous) functions $M \to K$ and that we need to use a larger class of functions, *semi-definable coordinate functions* $\phi : M \to K$. We introduce a K-vector space $\mathcal{H}$ generated by these functions and define linear operators on $\mathcal{H}$ corresponding to the actions by $\tilde{G}$. These generate a non-commutative K-algebra A on $\mathcal{H}$. Importantly, A is determined uniquely (up to the choice of the language) in spite of the fact that $\mathcal{H}$ is not. Also, a non-trivial, semi-definable function induces on K some extra structure, which we call here $*$-data. Correspondingly, this adds some extra structure to the K-algebra A, which eventually makes it a C^*-algebra. Finally, we are able to recover the M from A. Namely, M is identified with the set of eigenspaces of 'self-adjoint' operators of A with the Zariski topology given by certain ideals of A. In other words, this new and more general class of Zariski geometries can be appropriately explained in terms of non-commutative co-ordinate rings.

We then discuss further links to non-commutative geometry. We show how, given a typical *quantum algebra A at roots of unity*, one can associate a Zariski geometry with A. This is similar to, although slightly different from, the connection between M and A in the preceding discussion. Importantly, for a typical non-commutative A, the geometry turns out to be non-classical, whereas for a commutative one, it is equivalent to the classical affine variety Max A.

The final chapter introduces a generalisation of the notion of a Zariski structure. We call the more general structures *analytic Zariski*. The main difference is that we no longer assume the Noetherianity of the topology. This makes the definition more complicated because we now have to distinguish between general closed subsets of M^n and the ones with better properties, which we call analytic. The main reward for the generalisation is that now we have a much wider class of classical structures (e.g. universal covers of some algebraic varieties) that satisfy the definition. One hope (which has not been realised so far) is to find a way to associate an analytic Zariski geometry with a generic quantum algebra.

The theory of analytic Zariski geometries is still in its infancy. We do not know if the algebraicity conjecture is true for analytic Zariski groups, which is an interesting and important problem. One of the main results presented here is the theorem stating that any compact, analytic Zariski structure is Noetherian, that is, it satisfies the basic definition. We also prove some model-theoretic properties of analytic Zariski structures, establishing their high level in the logical hierarchy, but remarkably this is the non-elementary logic stability hierarchy formulated in terms of Shelah's *abstract elementary classes*. This is a relatively new domain of model theory, and analytic Zariski structures constitute a large class of examples for this theory.

We hope that these notes may be useful not only for model theorists but also for people who have a more classical, geometric background. For this reason, we start the notes with a crash course in model theory. It is really basic, and the most important thing to learn in this section is the spirit of model theory. The emphasis on the study of definability with respect to a formal language is perhaps central to doing mathematics in a model-theoretic way.

1.2 About model theory

This section gives a very basic overview of model-theoretic notions and methods. We hope that the reader will be able to grasp the main ideas and the spirit of the subject. We did not aim to give proofs in this section of every statement we found useful, and even definitions are missing some detail. To compensate for this, in Appendix A we give a detailed list of basic model-theoretic facts, definitions, and proofs. Appendix B surveys geometric stability theory and some more recent results relevant to the material in the main chapters.

Of course, there is a good selection of textbooks on model theory. The most adequate for our purposes is Marker (2002), and a more universal book is Hodges (1993).

The crucial feature of the model-theoretic approach to mathematics is the attention paid to the formalism with which one considers particular mathematical structures.

A structure M is given by a set M, the **universe** (or the **domain**) of M, and a family L of relations on M, called **primitives of** L or **basic relations**. One often writes $\mathrm{M} = (M, L)$. L is called the **language** for M.

Each relation has a fixed name and arity, which allows us to consider classes of L-structures of the form (N, L), where N is a universe and L is the collection of relations on N with the names and arities fixed (by L). Each such structure (N, L) represents an **interpretation** of the language L.

Recall that an n-ary relation S on M can be identified with a subset $S \subseteq M^n$. When S is just a singleton $\{s\}$, the name for S is often called a **constant symbol** of the language. One can also express functions in terms of relations; instead of saying $f(x_1, \ldots, x_n) = y$, one says just that $\langle x_1, \ldots, x_n, y\rangle$ satisfies the $(n+1)$-ary relation $f(x_1, \ldots, x_n) = y$. There is no need to include special function and constant symbols in L.

One always assumes that the binary relation $=$ is in the language and is interpreted canonically.

Definition 1.2.1. The following is an inductive definition of a **definable set** in an L-structure M:

(i) a set $S \subseteq M^n$ interpreting a primitive S of the language L is definable;
(ii) given definable $S_1 \subseteq M^n$ and $S_2 \subseteq M^m$, the set $S_1 \times S_2 \subseteq M^{n+m}$ is definable (here $S_1 \times S_2 = \{x^\frown y : x \in S_1,\ y \in S_2\}$);

(iii) given definable $S_1, S_2 \subseteq M^n$, the sets $S_1 \cap S_2$, $S_1 \cup S_2$, and $M^n \setminus S_1$ are definable; and
(iv) given definable $S \subseteq M^n$ and a projection $\mathrm{pr} : \langle x_1, \ldots, x_n\rangle \mapsto \langle x_{i_1}, \ldots, x_{i_m}\rangle$, $\mathrm{pr} : M^n \to M^m$, the image $\mathrm{pr}\, S \subseteq M^m$ is definable.

Note that item (iv), for $n = m$, allows a permutation of variables.

The definition can also be applied to definable functions, definable relations, and even definable points.

An alternative but equivalent definition is given by introducing the (first-order) L-formulas. In this approach, we write $S(x_1, \ldots, x_n)$ instead of $\langle x_1, \ldots, x_n\rangle \in S$, starting from basic relations, and then we construct arbitrary formulas by induction using the logical connectives $\wedge$, $\vee$, and $\neg$ and the quantifier $\exists$.

Now, given an L-formula ψ with n free variables, the set of the form

$$\psi(M^n) := \{\langle x_1, \ldots, x_n\rangle \in M^n : \mathrm{M} \vDash \psi(x_1, \ldots, x_n)\},$$

is said to be definable (by formula ψ).

The approach via formulas is more flexible because we may use formulas to define sets with the same formal description, say $\psi(N^n)$, in arbitrary L-structures.

Moreover, if formula ψ has no free variables (then called a **sentence**), it describes a property of the structure itself. In this way, classes of L-structures can be defined by axioms in the form of L-formulas.

One says that N is **elementarily equivalent** to M (written $\mathrm{N} \equiv \mathrm{M}$) if for all L-sentences φ

$$\mathrm{M} \vDash \varphi \Leftrightarrow \mathrm{N} \vDash \varphi.$$

Example 1.2.2. Groups can be considered L-structures where L has one constant symbol e and one ternary relation symbol $P(x, y, z)$, interpreted as $x \cdot y = z$. For example, the associativity property then can be written as

$$\forall x, y, z, u, v, w, t\ \ (P(x, y, u) \wedge P(u, z, v) \wedge P(x, w, t) \wedge P(y, z, w) \rightarrow v = t).$$

Here $\forall x\, A$ means $\neg\exists x\, \neg A$, and the meaning of $B \to C$ is $\neg B \vee C$.

The centre of a group G can be defined as $\varphi(G)$, where $\varphi(x)$ is the formula

$$\forall y, z\, (P(x, y, z) \leftrightarrow P(y, x, z)).$$

Of course, this definable set can be described in line with Definition 1.2.1, although this would be slightly longer description.

One important advantage of Definition 1.2.1 is that it provides a more geometric description of the set. We use both approaches interchangeably.

One of the most useful types of model-theoretic results is a **quantifier elimination** statement. One says that M (or more usually, the theory of M) has quantifier elimination if any definable set $S \subseteq M^n$ is of the form $S = \psi(M^n)$ where $\psi(\bar{x})$ is a **quantifier-free** formula, that is, one obtained from primitives of the language using connectives but no quantifiers.

Example 1.2.3. Define the language L_{Zar} with primitives given by zero-sets of polynomials over the prime subfield.

Theorem (Tarski, also Seidenberg and Chevalley). *An algebraically closed field has quantifier elimination in language* L_{Zar}.

Recall that in algebraic geometry, a Boolean combination of zero-sets of polynomials (Zariski closed sets) is called a constructible set. The theorem says, in other words, that the class of definable sets in an algebraically closed field is the same as the class of constructible sets.

Note that for each S, the fact that $S = \psi(M^n)$ is expressible by the L-sentence $\forall \bar{x}(S(\bar{x}) \leftrightarrow \psi(\bar{x}))$. Hence, quantifier elimination holds in M if and only if it holds in any structure elementarily equivalent to M.

Given a class of elementarily equivalent L-structures, the adequate notion of embedding is that of an **elementary embedding**. We say that $\mathrm{M} = (M, L)$ is an elementary substructure of $\mathrm{M}' = (M', L)$ if $M \subseteq M'$ and for any L-formula $\psi(\bar{x})$ with free variables $\bar{x} = \langle x_1, \ldots, x_n \rangle$ and any $\bar{a} \in M^n$,

$$\mathrm{M} \vDash \psi(\bar{a}) \Leftrightarrow \mathrm{M}' \vDash \psi(\bar{a}).$$

More generally, elementary embedding of M into M′ means that M is isomorphic to an elementary substructure of M′. We write the elementary embedding (elementary extension) as

$$\mathrm{M} \preccurlyeq \mathrm{M}'.$$

Note that $\mathrm{M} \preccurlyeq \mathrm{M}'$ always implies that $\mathrm{M} \equiv \mathrm{M}'$, because an elementary embedding preserves all L-formulas, including sentences.

Example 1.2.4. Let $\mathbb{Z}$ be the additive group of integers in the group language of Example 1.2.2. Obviously $z \mapsto 2z$ embeds $\mathbb{Z}$ into itself as $2\mathbb{Z}$. However, this is not an elementary embedding because the formula $\exists y\, y + y = x$ holds for $x = 2$ in $\mathbb{Z}$ but does not hold in the substructure $2\mathbb{Z}$. On the other hand, for

$K \subseteq K'$ algebraically closed fields in language L_{Zar}, the embedding is always elementary. This is an immediate result of the quantifier elimination theorem.

A simple but useful technical fact is given by the following.

Exercise 1.2.5. *Let*

$$\mathrm{M}_1 \prec \cdots \mathrm{M}_\alpha \prec \mathrm{M}_{\alpha+1} \prec \cdots$$

be an ascending sequence of elementary extensions, $\alpha \in I$, *and let*

$$^*\mathrm{M} = \bigcup_{\alpha \in I} \mathrm{M}_\alpha$$

be the union. Then, for each $\alpha \in I$, $\mathrm{M}_\alpha \prec {}^*\mathrm{M}$.

When we want to specify an element in a structure M in terms of L, we describe its **type**. Given $\bar{a} \in M^n$, the **type** of $\bar{a}$ is the set of L-formulas with n free variables $\bar{x}$:

$$\mathrm{tp}(\bar{a}) = \{\psi(\bar{x}) : \ \mathrm{M} \vDash \psi(\bar{a})\}.$$

Often we look for n-tuples, in M or its elementary extensions, that satisfy a certain description in terms of L. For this purpose, one uses a more general notion of a type.

Definition 1.2.6. An n-type in M is a set p of L-formulas $\psi(\bar{x})$ (with free variables $\bar{x} = \langle x_1, \ldots, x_n \rangle$) satisfying the consistency condition:

$$\psi_1(\bar{x}), \ldots, \psi_k(\bar{x}) \in p \Rightarrow \mathrm{M} \vDash \exists \bar{x}\, \psi_1(\bar{x}) \wedge \cdots \wedge \psi_k(\bar{x}).$$

Obviously, the M in the consistency condition can be equivalently replaced by any M′ elementarily equivalent to M.

Example 1.2.7. Let $\mathbb{R}$ be the field of reals in language L_{Zar}. Note that the relation $x \leq y$ is expressible in $\mathbb{R}$ by the formula $\exists u\, u^2 + x = y$. So, in the language we can write down the type of a real positive **infinitesimal**,

$$p = \left\{ 0 < x < \frac{1}{n} : \ n \in \mathbb{Z}, n > 0 \right\}.$$

Obviously, this type is not realised in $\mathbb{R}$ itself, but there is $\mathbb{R}' \succ \mathbb{R}$, which realises p.

Often, we have to consider L-formulas **with parameters**. For example, in Example 1.2.3, the basic relations are given by polynomial equations over the prime field, but one usually is interested in polynomial equations over K. Clearly, this can be achieved within the same language if we use parameters: if $P(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a polynomial equation over the prime field and