Part I

Lecture Notes

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Lectures on Principal Bundles

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To Peter Newstead on his 65th birthday

1 Introduction

The aim of these lectures (†) is to give a brief introduction to principal bundles on algebraic curves towards the construction of the moduli spaces of semistable principal bundles. The second section develops the basic machinery on principal bundles and their automorphisms. At the end of the second section, we give a proof of theorem of Grothendieck on orthogonal bundles. The third section, after developing the notions of semistability and stability, gives a modern proof of the main part of Grothendieck's theorem on classification of principal bundles on the projective line. The last section gives an outline of the construction of the moduli space of principal bundles on curves. The moduli space was constructed by A.Ramanthan in 1976. The method outlined here is from a construction in [BS]. These notes are a transcription of the lectures given in Mexico and therefore have an air of informality about them. I have consciously retained this informality despite criticism from a learned referee on the lack of rigor in some places. Indeed(‡) "*these notes are almost exactly in the form in which they were first written and distributed: as class notes, supplementing and working out my oral lectures. As such, they are far from polished and ask a lot of the reader. ...Be that as it may, my hope is that a well-intentioned reader will still be able to penetrate these notes and learn something of the subject*".

† These are notes of five lectures given in Mexico in November 2006 at CIMAT, Guanajuato in the "College on Vector Bundles" which was held in honour of Peter Newstead to celebrate his 65^{th} birthday.
‡ Lifted from D.Mumford's introduction to his "Lectures on curves on an algebraic surface"

2 Basic notions and definitions

Throughout these notes, unless otherwise stated, we have the following notations and assumptions:

(a) We work over an algebraically closed field k of characteristic zero and without loss of generality we can take k to be the field of complex numbers $\mathbb{C}$.
(b) G will stand for a reductive algebraic group often the general linear group $GL(n)$ and H a subgroup of G. Their representations are finite dimensional and rational.
(c) X is a smooth projective curve almost always in these notes.

2.1 Generalities on principal bundles

Definition 2.1. A principal G bundle $\pi : E \longrightarrow X$ with structure group G (or a G-bundle for short) is a variety E with a right G-action, the action being free, such that π is G-equivariant, X being given the trivial action. Further, the bundle π is locally trivial in the étale topology. (In other words, for every $x \in X$, there exists a neighbourhood U and an étale morphism $U' \longrightarrow U$ such that, when E is pulled back to U', it is *trivial* as G-bundle).

Remark 2.2. We remark that in our setting this definition of a principal bundle is related to the definition of a principal bundle being *locally isotrivial*, ([S]) i.e. for every $x \in X$, there exists a neighbourhood U and a finite and unramified morphism $U' \longrightarrow U$ such that, when E is pulled back to U', it is *trivial* as G-bundle. In fact, these two definitions coincide here because of the affineness of the group G. One could see this as follows: since G is affine, we have an inclusion of G in $GL(V)$ as a closed subgroup. This implies that $GL(V) \longrightarrow GL(V)/G$ is *locally isotrivial* in the sense of Serre ([S]). We now observe that any principal G–bundle can be obtained from a principal $GL(V)$–bundle by a *reduction of structure group* (see (2.5)). Further, any principal $GL(V)$–bundle is actually locally trivial in the Zariski topology (see [S]). The local isotriviality of the G–bundle now follows from that of the quotient map $GL(V) \to GL(V)/G$. In this context cf ([M]).

Remark 2.3. In general in the literature, a principal homogeneous space is defined to be locally trivial in the so-called fppf topology if the base is not smooth. Again, we remark that in our setting this is not needed even when the base is not smooth because we work with affine groups as structure groups. This follows from arguments similar to the one given above since any principal homogeneous space for the group $GL(V)$ is locally trivial in the Zariski topology.

2.1.1 Some notations and conventions

(a) By a family of H bundles on X parametrised by T we mean a principal H-bundle on $X \times T$, which we also denote by $\{E_t\}_{t\in T}$. We note that in general we may not have T to be smooth and the definition of principal bundles should be seen in the light of Remark 2.3.

(b) We recall the definitions of semisimple and reductive algebraic groups. Given a linear algebraic group G, we define the radical $R(G)$ to be $(\bigcap B)^0$ where the intersection runs over all Borel subgroups. Equivalently, $R(G)$ is the maximal, *normal*, connected, solvable subgroup of G. If $R(G) = (e)$ then G is called *semisimple*, and if $R_u(G)$, the unipotent radical of G which consists of the unipotent elements in $R(G)$, is trivial, then G is called *reductive*. (In this case $R(G)$ will be a torus). Equivalently, (by considering the derived subgroup), G is *semisimple* (resp. *reductive*) if and only if it has no connected abelian (resp. unipotent abelian), normal subgroup other than (e).

(c) Let Y be any quasi projective G-variety and let E be a G-principal bundle. For example Y could be a G-module. Then we denote by $E(Y)$ the associated bundle with fibre type Y which is the following object: $E(Y) = (E \times Y)/G$ for the twisted action of G on $E \times Y$ given by $g.(e, y) = (e.g, g^{-1}.y)$.

(d) Any G-equivariant map $\phi : F_1 \longrightarrow F_2$ will induce a morphism $E(\phi) : E(F_1) \longrightarrow E(F_2)$.

(e) A section $s : X \longrightarrow E(F)$ is given by a morphism

$$s' : E \longrightarrow F$$

such that, $s'(e.g) = g^{-1}.s'(e)$ and $s(x) = (e, s'(e))$, where $e \in E$ is such that $\pi(e) = x$, where $\pi : E \longrightarrow X$.

Definition 2.4. If $\rho : H \longrightarrow G$ is a homomorphism of groups the associated bundle $E(G)$, for the action of H on G by left multiplication through ρ, is naturally a G–bundle. We denote this G–bundle often by $\rho_*(E)$ and we say this bundle is obtained from E by extension of structure group.

Definition 2.5. A pair (E, ϕ), where E is an H–bundle and $\phi : E(G) \longrightarrow F$ is a G–bundle isomorphism, is said to give a reduction of structure group of the bundle F to H. For convenience, we often omit ϕ and simply say E is obtained from F by reduction of structure group.

Two H–reductions of structure group (E_1, ϕ_1) and (E_2, ϕ_2) are equivalent or isomorphic if there is an H–bundle isomorphism $f : E_1 \longrightarrow E_2$ such that the following diagram commutes:

$$\begin{tikzcd} E_1(G) \arrow[r, "f(G)"'] \arrow[d, "\phi_1"'] & E_2(G) \arrow[d, "\phi_2"'] \\ F \arrow[r, "="'] & F \end{tikzcd} \tag{1.1}$$

Remark 2.6. A principal G-bundle E on X has an H-structure or equivalently a reduction of structure group to H if we are given a section $\sigma : X \longrightarrow E(G/H)$, where $E(G/H) \simeq E \times^G G/H$. To see this, we note the identification of the spaces $E(G/H) \simeq E/H$. Then by pulling back the principal H–bundle $E \longrightarrow E/H$ by σ, we get an H–bundle $E_H \subset E$, giving the required H-reduction. In other words, there is a natural isomorphism $E_H(G) \simeq E$. Thus, we get a correspondence between sections of $E(G/H)$ and H–reductions of E.

To see the other direction, let E_H be an H–bundle and consider the natural inclusion $E_H \hookrightarrow E_H(G) = E$ given by $z \longrightarrow (z, 1_G)$, where of course $(z, 1_G)$ is identified with (zh, h^{-1}) for $h \in H$. Going down by an action of H we get a map $X \longrightarrow E_H(G)/H = E(G/H)$. This gives the required section of $E(G/H)$.

Remark 2.7. Note that a reduction of structure group of E to $H \subset G$ can be realised by giving a G-map $s : E \longrightarrow G/H$ satisfying the property $s(e.g) = g^{-1}s(e)$. In this sense, the reduction E_H defined above can be seen to be the inverse image of the identity coset in G/H by the map s.

Remark 2.8. In the case of $G = GL(n)$, when we speak of a principal G-bundle we identify it often with the associated vector bundle by taking the associated vector bundle for the standard representation.

Remark 2.9. A $GL(n)$–bundle is completely determined by the associated vector bundle $E(V)$ (where V is the canonical n-dimensional space on which $GL(n)$ acts) as its bundle of frames. Let $\mathcal{V}$ be a vector bundle on X with fibre the vector space V. Then consider the union

$$\bigcup_{x \in X} Isom(V, \mathcal{V}_x)$$

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where $Isom(V, \mathcal{V}_x)$ are simply isomorphisms between the vector spaces V and $\mathcal{V}_x$.

Note that there is a natural action of $GL(V)$ on the right which is easily seen to be free. This forms the total space of the principal $GL(V)$–bundle E whose associated vector bundle $E(V)$ is isomorphic to $\mathcal{V}$.

Similarly, a $PGL(n)$–bundle is equivalent to a projective bundle, i.e. an isotrivial bundle with $\mathbf{P^n}$ as fibre.

Proposition 2.10. Let E_1 and E_2 be two H–bundles. Giving an isomorphism of the H–bundles E_i is equivalent to giving a reduction of structure group of the principal $H \times H$–bundle $E_1 \times_X E_2$ to the diagonal subgroup $\Delta_H \subset H \times H$.

Proof: A reduction of structure group to the diagonal Δ gives a Δ–bundle E_Δ. Now observe that the projection maps on $H \times H$, when restricted to the diagonal, give isomorphisms of $\Delta \simeq H$. Viewing the bundle $E_\Delta \subset E_1 \times_X E_2$ as included in the fibre product, and using the two projections to E_1 and E_2, we get isomorphisms from $E_1 \simeq E_\Delta \simeq E_2$. The converse is left as an exercise.

Definition 2.11. Let P be a G–bundle. Consider the canonical adjoint action of G on itself, i.e $g \cdot g' = gg'g^{-1}$. Then we denote the associated bundle $P \times^G G$ by Ad(P).

Observe that because of the presence of an identity section, the associated fibration $Ad(P)$ is in fact a group scheme over X.

Proposition 2.12. The sections $\Gamma(X, Ad(P))$ are precisely the G–bundle automorphisms of P.

Proof: Let $\sigma : X \longrightarrow Ad(P)$ be a section. We view the section σ as remarked above as an equivariant map $\sigma : P \longrightarrow G$. Then by its definition, we have the following equivariance relation:

$$\sigma(p.g) = g^{-1} \cdot \sigma(p) = g^{-1}\sigma(p)g$$

Define the morphism:

$$f_\sigma : P \longrightarrow P$$

given by

$$f_\sigma(p) = p.\sigma(p)$$

$\forall p \in P$. The equivariance property for σ implies that $f_\sigma(p.g) = f_\sigma(p).g$ and hence its an H–morphism. Clearly it gives an automorphism of P.

For the converse, let $f : P \longrightarrow P$ be an H–bundle automorphism. Define σ as follows:

$$f(p) = p.\sigma(p)$$

Note that the H–equivariance property of f implies the equivariance property of σ which therefore defines a section of $Ad(P)$.

Remark 2.13. Let E be a reduction of structure group of the principal G–bundle P to the subgroup H. Then as we have remarked above, we can represent E as a pair (P, ϕ) where $\phi : X \longrightarrow P(G/H)$ is a section of the associated fibration. Let σ be an automorphism of the G–bundle P. Then, σ also acts as an automorphism on the associated fibration $P(G/H)$. This gives an action of the group $Aut(P)$ on the *set of all H–reductions of P*.

Two H–reductions E and F of a principal G–bundle are equivalent (i.e give isomorphic H–bundles) if and only if there exists an automorphism σ of P which takes E to F in the above sense.

To illustrate this phenomenon I give below a theorem due to Grothendieck ([G]).

Theorem 2.14. Let X be a smooth projective complex variety and let $H = O(n) \subset G = GL(n)$ be the standard inclusion. Then the canonical map induced by extension of structure groups

$$\{Isom\ classes\ of\ H-bundles\} \longrightarrow \{Isom\ classes\ of\ G-bundles\}$$

is injective. In other words, a G–bundle P has, if any, a unique reduction of structure group to H upto equivalence.

Proof: The proof of this theorem is quite beautiful and I will give it in full. Its also of importance to observe that the theorem is false for the inclusion $SO(n) \subset SL(n)$!

Let S be the space of symmetric $n \times n$–matrices which are non-singular. Then G acts on S as follows:

$$A.X := AXA^t$$

The action is known (by the Spectral theorem for non-degenerate quadratic forms) to be transitive and the isotropy subgroup at I is the standard orthogonal group $H = O(n)$. i.e $S \simeq G/H$ as a G–space. The more important fact is that there is a canonical inclusion of S in G. If $q : G \longrightarrow G/H$ is the canonical quotient map then identifying the quotient with S, the map q is given by

$$q(A) = AA^t$$

and then the restriction of the map to $S \hookrightarrow G$ is given by the map $q_S(A) = A^2$ on the space of symmetric matrices.

$$\begin{tikzcd} S \arrow[r, "\text{inclusion}"'] \arrow[d, "q_S"'] & G \arrow[d, "q"] \\ S \arrow[r, "="'] & G/H \end{tikzcd} \tag{1.2}$$

Let P_H be a fixed reduction of structure group of P to H and let E be any other reduction of P which is therefore given by a section $\sigma : X \longrightarrow P(G/H)$. Since we already have a reduction, we can express the new reduction σ as:

$$\sigma : X \longrightarrow P_H(G/H)$$

Consider the group scheme $Ad(P) = P(G)$. Since P has a H–reduction, we can view this group scheme as $P_H(G)$, where $H \hookrightarrow G$ acts on G by conjugation i.e $h \cdot g = h.g.h^{-1}$.

We also observe that the associated fibration $P_H(G/H)$ taken with the natural left action of H on G via its inclusion or by the conjugation action of H on G is identical. In other words, we can view the morphism associated to the canonical quotient map $G \longrightarrow G/H$ for the associated fibrations $P_H(G)$ and $P_H(G/H)$ as being induced by the conjugation action of H on G and G/H. Note that this is special to our situation since we have a H–reduction P_H to start with.

We thus get the map:

$$\phi : P_H(G) \longrightarrow P_H(G/H)$$

Observe again that the inclusion $S \hookrightarrow G$ is an H–morphism for the conjugation action of H. (Since $H = O(n)$, $A^t = A^{-1}$).

Viewing the spaces in the diagram above as a diagram of H–spaces for the conjugation action we have the following diagram of associated spaces:

$$\begin{tikzcd} P_H(S) \arrow[r, "\text{inclusion}"'] \arrow[d, "q_S"'] & P_H(G) \arrow[d, "\phi"] \\ P_H(S) \arrow[r, "="'] & P_H(G/H) \end{tikzcd} \tag{1.3}$$

We now note that to prove that the reduction of P given by σ is equivalent to the one given by P_H, we need to give an automorphism which takes one to the other. An automorphism is giving a section of $P(G)$ or equivalently of $P_H(G)$. Its easy to check that, giving such an automorphism is giving a section $\gamma : X \longrightarrow P_H(G)$ such that the following diagram commutes:

$$\begin{tikzcd} P_H(G) \arrow[r, "\phi"] & P_H(G/H) \\ X \arrow[u, "\gamma"] \arrow[ur, "\sigma"] & \end{tikzcd} \tag{1.4}$$

We now recall the following *interpolation* statement:

Lemma 2.15. The characteristic polynomial of a non-singular matrix A can be used to get a square root of A.

Proof: By an interpolation exercise, we can construct a polynomial $h(t)$ such that $h(t)^2 - t$ is divisible by $f(t)$, i.e $h^2(t) - t = f(t)g(t)$ for some polynomial $g(t)$. Since $f(A) = 0$ by the Cayley-Hamilton theorem, we get $h(A)^2 = A$, i.e. $h(A)$ provides a square root of A.

Now identify $P_H(G/H)$ with $P_H(S)$, which is a bundle of non-singular symmetric matrices. The section σ gives a characteristic polynomial $f(t)$ with coefficients being regular functions on X. Since X is projective these coefficients are therefore *constant*. Since these coefficients are constant, we can use the characteristic polynomial to get the *square root* of the section σ.

Take $h(t)$ as above. Then define

$$\eta(x) := h(\sigma(x))$$

This provides a section of $P_H(S)$ such that $\phi \circ \eta = \sigma$ since ϕ on S is the *squaring* operation. The composition $\eta : X \longrightarrow P_H(S) \hookrightarrow P_H(G)$ gives the required γ.

Remark 2.16. The reader should try and understand the proof in Grothendieck's paper. The main idea, as suggested by Prof Ramanan, is to compare a pair of equivalent non-degenerate quadratic forms on a vector space. When this is carried out over a family, together with choosing a square root (over X) it is essentially the classical proof. The reader can note that the proof given above works for all characteristics different from 2. The proof given here applies the definitions developed here and also naturally generalises the problem in the following sense.

In general one could work with a symmetric space G/H, where the subgroup H is given by the fixed points of an involution on G. For example, the proof generalises immediately to the case of the symplectic group. It also suggests that one could look for natural conditions for the map to be an inclusion.

Remark 2.17. It fails for $SO(n) \subset SL(n)$. In fact, it fails for $n = 2$. This can be seen as follows: $SO(2) \simeq \mathbf{G}_m$. Hence $SO(2)$ principal bundles can be identified with $\mathbf{G}_m$-bundles and hence with line bundles. Extension of structure group of a $SO(2)$–bundle to $SL(2)$ is equivalent to taking a line bundle L to $L \oplus L^*$, which is an $SL(2)$–bundle. This has clearly two inequivalent reductions L and L^*.

Remark 2.18. Olivier Serman ([O]) has shown that the map studied in the above theorem actually extends to an embedding of the moduli spaces of S–equivalence class of semistable orthogonal principal bundles into the moduli space of semistable principal $GL(n)$–bundles (see Section 4 below for the definitions).

3 Principal bundles, basic properties

Definition 3.1. A vector bundle V is said to be *semistable* (resp *stable*) if for every sub-bundle $W \subset V$,

$$\frac{deg(W)}{rk(W)} \le \frac{deg(V)}{rk(V)}.$$

Lemma 3.2. Let V and W be semistable vector bundles on X of degree zero. Then $V \otimes W$ is semistable of degree zero.

Proof. Any semistable bundle on X of degree zero has a Jordan-Holder filtration such that its associated graded is a direct sum of stable bundles of degree zero. Note that the filtration is not unique but the associated graded is so. Hence the tensor product $V \otimes W$ gets a filtration such that its associated graded is a direct sum of tensor products of stable bundles of degree zero. We see easily that this reduces to proving the lemma when V and W are stable of degree zero. Then by the Narasimhan-Seshadri theorem, $V \otimes W$ is defined by a unitary representation of the fundamental group (namely the tensor product of the irreducible unitary representations which define V and W respectively), which implies that $V \otimes W$ is semistable (see [NS]).