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Edited by Frederick P. Gardiner, Gabino Gonzalez-Diez and Christos Kourouniotis
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Geometry of Riemann Surfaces

Proceedings of the Anogia Conference to celebrate
the 65th birthday of William J. Harvey

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Preface

This conference on the Geometry of Riemann Surfaces and related topics was held in the beautiful hill town of Anogia at the Conference Centre of the University of Crete, spanning four days in June and July 2007. The pretext was the celebration of Bill Harvey's 65th birthday and retirement from teaching. About 50 mathematicians and friends came, many from far-flung points of the globe, to enjoy this opportunity to refresh mind, body and spirit.

We invited all participants to contribute articles based on their talks or related material; the response was wholehearted and expert, with the result that you see before you. The broad range of topics addressed by the articles reflects the pervasive influence of the theory of Riemann surfaces and the remarkable variety of geometric ideas and methods which flow from it; this expansive aspect of the field will be discussed by Bill Harvey in the introduction which follows.

We take this opportunity also to thank Professor Harvey himself, whose supportive and knowledgeable comments provided foil and inspiration for all the participants.

On behalf of those lucky enough to be at Anogia for this conference, we thank all the sponsors, the Universidad Autónoma de Madrid and Comunidad Autónoma de Madrid (Grant C-101), the Spanish Government Ministerio Español de Educación y Ciencia (Grants MTM2006-01859 & MTM2006-28257-E), the Department of Mathematics of the University of Crete and the Anogia Academic Village, Crete, for generous financial support which made the meeting possible; in particular, the funding provided subsistence and travel expenses for graduate students and others lacking support.

We are also very grateful to the local organisers and staff at the University of Crete, in Iraklio and at the Conference Centre, and especially the conference secretary Marina Vasilaki, for their work in planning and preparation and for friendly assistance during the meeting. Significant editorial assistance with the Proceedings was given by David Torres Teigell (UAM) and is gratefully acknowledged. Finally, the appearance of this volume is thanks to the unselfish hard work of many people including all the anonymous referees, testament in itself to the continuing good health of this section of the mathematical community.

Fred Gardiner
Gabino González-Diez
Christos Kourouniotis

Foreword: Riemann surfaces and a little history.

William Harvey,
King's College London

A Riemann surface is a thing of beauty, possessing geometric shape as well as analytic or algebraic structure. From its introduction in 1851 in Riemann's inaugural dissertation, his first great work establishing the foundations of geometric complex analysis, the concept has exerted an unusual influence as a powerful clarifying mental tool.

Today, the pervasive role of complex analysis in the mathematical and physical sciences has brought these ideas into a significance wider than even their founder could have predicted. In the present book, the reader will find a selection of results which can only indicate the part currently played by surfaces and their spaces of deformations: just as a single convergent power series is enough to generate by continuation an entire Riemann surface structure, so the foundational ideas of our discipline extend and evolve beyond our present view of them.

Central to the contemporary study of Riemann surfaces is the interplay between different aspects, geometric ideas and algebraic or analytical calculations, leading to insights into the deeper properties these objects possess. The basic notion provides a topological base for deploying the most powerful ideas of algebra, geometry and analysis: indeed it establishes a central role for topology in bringing about a unique mathematical synthesis. A single accessible theory serves to interconnect complex analysis and the various algebraic invariants, the fundamental group, field of functions, homology and period lattices. In the reassuring familiarity of a two dimensional framework, we have a global base for complex analytic and covering space methods, interacting with Galois-theoretic properties of the function field. And awaiting new developments, there are so many areas where Riemann surfaces are direct contributors: hyperbolic manifolds and kleinian groups, iteration of polynomial or holomorphic functions, crystallographic groups, geometric group theory, conformal quantum field theory and symplectic geometry. Not least, there is the burgeoning field of complex geometry, which relies on the theory of Riemann surfaces for insights and testing of ideas as well as for essential tools in the study of complex or symplectic manifolds, via deformation theory and the existence of embedded curves. All these areas are represented among the articles included here, a blend of original research, broad surveys and applications, as indicated in the next section.

1 Background and development.

The brief historical narrative which follows will (it is hoped) set the scene and draw together all the diverse themes to be found in this volume. Much of it relies, directly or indirectly, on the mapping theorem of Riemann and its successor, the uniformisation theorem of Poincaré and Koebe, which deliver an intrinsic geometric structure by covering projection from the unit disc to any Riemann surface or complex algebraic curve, casting new light on purely analytic or algebraic matters. Thus, as an instance of how important insights follow, any closed loop not bounding a disc on a hyperbolic surface S determines (by the calculus of variations) a unique closed geodesic in the same free homotopy class, whose length is an important geometric invariant, an element of the *length spectrum* of that surface. This discrete collection of positive numbers encodes the conformal shape in an extremely subtle way, related via famous work by A. Selberg to the analytic study of the Laplace-Beltrami operator, a global elliptic operator on $\mathcal{L}_2(S)$, whose spectral properties are of wide interest. Taking a different line from the same beginning, such a loop in S determines a smooth real *length function* on the moduli space of conformal classes of hyperbolic surfaces diffeomorphic to S, which is part of a Morse-theoretic topological decomposition of the moduli space. Three articles in the book pursue this theme. In Peter Buser's article, an algorithmic approach is given to the question of computing the set of shortest geodesics from a suitable specification of the (marked) surface's moduli and carried out for a specific example, a breakthrough in explicit geometric computation on a surface, with roots in work by Poincaré and Dehn on algorithms for simple loops. The paper by Greg McShane and Hugo Parlier addresses the issue of multiplicities for the length spectrum for simple loops on a (punctured) torus, a classic test case for hyperbolic surface phenomena. Robert Silhol's paper addresses a quite different issue in this geometric arena, which has bearing on the coefficient field of the uniformising fuchsian group.

Over the past sixty years, Riemann surface theory has seen a progression of new ideas and methods, none more influential than the development during the 1950s, by L.V. Ahlfors, L. Bers and H.E. Rauch, of a theory of moduli for surface deformations, holomorphic parameters for a family of surfaces of specified topological type. The case of the flat torus (genus 1) is classical, but for higher genus it was not until the foundational ideas of O. Teichmüller were digested and sharpened that Ahlfors and Bers could establish a comprehensive analytic theory of Riemann surface families based on *quasiconformal homeomorphisms*, or q-c maps. This type of controlled deformation has special flexibility and plays a prominent part in the study of holomorphic families of discrete groups and dynamical systems in complex dimension 1. The crucial first step was the construction of a complex analytic Teichmüller space of moduli for a given compact Riemann surface or orbifold, a parameter space for all holomorphic deformations of fixed type carrying a given topological marking; this was established using q-c maps soon after 1960. The complex analytic and Kähler metric structure of the moduli spaces $\mathcal{M}_g$ of conformal classes for surfaces of genus $g \geq 2$ soon ensued and attention then focussed on the construction of a suitable compactification within the parallel development of q-c deformations for all finitely generated kleinian groups – these are discrete groups of Möbius transformations which occur as cover transformations in uniformisation of surfaces when the covering region is not necessarily a Euclidean disc.

A very different approach to moduli of curves was completed by D. Mumford around 1965, using geometric invariant theory, the algebraic geometer's approach to classification problems; later with P. Deligne he extended his earlier results with A. Mayer on stable degeneration of curves by acquisition of double points to produce a projective completion $\widehat{\mathcal{M}}_g$, the algebraic variety of moduli of stable curves of fixed genus $g \geq 2$. The noded 'stable curves' which fill up the missing boundary divisor represent cusps of the corresponding mapping class group action within a complex analytic compactification, analogous to the point at infinity, and its orbit under the modular group $\mathrm{SL}_2(\mathbb{Z})$, in the space of marked tori. This chimed with the detailed classification by Bers, by B. Maskit and others of regular b-groups, a particular kind of kleinian group which represent noded curves within a kind of local completion of the familar spaces of fuchsian and quasifuchsian groups; this too gives a compactification of complex moduli space, isomorphic in an appropriate sense to that of Deligne and Mumford. A theory of deformations for all projective complex manifolds had been developed earlier by K. Kodaira and D.C. Spencer and by M. Kuranishi, but the special nature of geometry in one complex dimension has created a vastly different, more detailed landscape. This was enhanced after 1980 by S. A. Wolpert's penetrating work on the Weil-Petersson metric structure, linking it to the length spectrum and hyperbolic surface geometry.

Around 1970, H.E. Rauch and his student H.M. Farkas were considering the *Schottky problem*, which asks for a precise characterisation of the place of curves and their associated Jacobi varieties – these are complex tori of dimension g which emerge by integrating a basis of holomorphic 1-forms along a set of generating loops for the homology on a curve – within the general theory of (principally polarised) abelian varieties and their periods. This question received a remarkable reformulation as part of the theory of completely integrable systems of differential equations, resulting in T. Shiota's resolution of the problem in 1985; however, because this approach is very inexplicit, further study continues today, and the article here by Victor Gonzalez contributes to it for the case of curves with symmetry. The projective geometry of linear systems of divisors on $p-$gonal curves, analysed in the article by Gabino González-Díez, is also significant for the study of these special curves and their Jacobi varieties.

By a theorem of Hurwitz, any compact Riemann surface of genus g (at least 2) has finite automorphism group with order at most $84(g-1)$. These groups lift to discrete (infinite) covering group actions on the universal covering disc as fractional linear automorphisms, thus providing a valuable way to study surface automorphism groups first discovered by Klein and Fricke and exploited by A.M. Macbeath and his students in the 1960s. In work pursued later by David Singerman and co-workers, this approach was extended to the case of all symmetry groups by expanding the framework to involve Macbeath's classification of the nonorientable hyperbolic crystallographic groups and their Teichmüller spaces together with symmetries of real algebraic curves. Articles presenting results on this aspect of the theory include a foundational one by Clifford Earle and two joint papers, by Emilio Bujalance, Javier Cirré & Gregor Gromadski and by Antonio Costa & Milagros Izquierda. Also, the uniformisation of surfaces using Schottky groups – another classical technique, which uses free kleinian groups operating on an intermediate planar covering of the Riemann surface – is extended here by Rubén Hidalgo & Bernard Maskit to include reflection symmetry.

Low dimensional topology and complex analysis are old friends; mapping class groups and braid groups have been part of this relationship from the beginning with Hurwitz, Klein and Poincaré, followed by Max Dehn's foundational work on abstract discrete groups and J. Nielsen's geometric analysis of surface mapping classes in the 1930s, with later contributions by W. Fenchel and by a long German tradition of low-dimensional topologists. The link was greatly strengthened in the 1960s by the school established under Wilhelm Magnus at New York University, who developed a detailed theory of finitely presentable groups with strong geometric ingredients. There is a direct connection between kleinian groups and hyperbolic 3-manifold structures stemming from the fact that the Riemann sphere, on which these groups act, forms the boundary of a standard model of hyperbolic space: this was observed by Poincaré, but thereafter lay dormant until Ahlfors reignited the subject after 1960 with his finiteness theorem. Several preliminary developments using hyperbolic space began after 1970; A. Marden established some key properties of kleinian 3-manifold quotients and their deformation spaces, and with C.J. Earle applied this to develop a distinctive new approach to completion of Teichmüller moduli spaces. Meanwhile R. Riley was studying the group theory of hyperbolic knot complements and T. Jorgensen began a deeper exploration of the special geometric crystallography which underlies the structure of discrete hyperbolic groups in dimensions 2 and 3. This topic expanded greatly, bringing further geometric insight into the general area of discrete group actions in real and complex hyperbolic spaces. Several articles in this book belong to this tradition, for instance the one by Jane Gilman and Linda Keen on the combinatorial structure of generating sets for free Möbius groups and the article by John Parker and Yiannis Platis describing recent progress on a higher dimensional complex hyperbolic analogue of quasifuchsian groups.

These aspects of Riemann surfaces were surveyed in an Instructional Conference on Discrete Groups and Automorphic Functions at Cambridge (UK) in 1975, which aimed to promote the field more widely among a new generation of students. A year later, W.P. Thurston brought about a revolution in the theory of kleinian groups and 3-manifolds by the introduction of new ideas and methods from dynamics, differential geometry and foliation theory, formulating his vision of geometrisation for 3-manifolds, the 3D analogue of surface uniformisation, and setting out an ambitious new agenda for research in low-dimensional geometry. The effect was electric. The existing framework of researchers expanded to meet the challenge, with concentrations across the USA and Europe, and all signed up with enthusiasm to Thurston's geometric approach, seeking a synthesis with the complex analytic one. A decade of intense activity ensued, with an explosion of results and fresh initiatives. Dynamical properties of real quadratic maps were analysed by J. Milnor and Thurston; a fundamental dictionary between the twin theories of kleinian groups and conformal dynamics was established by D. Sullivan, bringing quasiconformal methods and renormalisation into the dynamics of iteration for rational maps of the Riemann sphere; the classification of critically finite rational maps by Thurston was followed by the theory of polynomial-like mappings due to A. Douady and J.H. Hubbard and their group at Paris-Sud (Orsay), who worked out a remarkable detailed analysis of the Mandelbrot family of quadratic polynomials. The insights thus gained into the role of quasiconformal deformations in conformal geometry, rational maps and fractal structures in the plane fed into C.T. McMullen's work, which emphatically expanded Sullivan's dictionary and established major aspects of the Thurston programme. They continue to bear fruit today. In particular, the concept of *holomorphic motion* of a closed planar set (such as the limit set of a kleinian group or Julia set of a rational map) introduced in a seminal paper of Mañé, Sad & Sullivan, and developed by Sullivan and Thurston and, independently, by Bers and Royden, has been very influential; the fundamental theorem that any holomorphic motion of a closed set extends to one of the whole Riemann sphere emerged in definitive form in later work by Slodkowski and by Chirka. There are two papers on this topic in the present volume: the first, by Frederick Gardiner, Yunping Jiang & Zhe Wang presents a new proof of the Slodkowski Theorem and a second, by Sudeb Mitra, surveys the applications to deformation theory. In addition, the article by Shaun Bullett describes a special blend of rational maps and kleinian groups to which these methods apply.

As if the arrival of Thurston on the scene were not enough upheaval, the end of the 1970s also marked the beginning of M. Gromov's comprehensive deconstruction and redesign of discrete group theory and hyperbolic differential geometry. This has seen the creation of fresh ways to distinguish types of group action on metric spaces, involving a flexible approach to curvature. Notably, Gromov introduced the concept of hyperbolic group, providing a unifying theme for a broad swath of results in topology involving groups which, like fuchsian and kleinian groups, act on a space of negative curvature. A new landscape emerged, now known as geometric group theory, subsuming the work of Dehn and

Magnus on combinatorial group theory. This is a broad-brush weakening of the standard approach to rigidity (which uses isometric conjugacy to compare group actions on spaces): familiar groups arising from geometric origins are taken as benchmarks for classification, while any discrete group is viewed as a geometric object in its own right by its action on a Cayley graph, and the classification of group actions on spaces is carried out using a new measure of distortion, called quasi-isometry. Martin Bridson's paper here focusses on the position of the mapping class group within this environment, bringing out in the process the significance of its action as the modular group of isometries of the Teichmüller space.

As a final comment concerning these articles, for me much of the excitement of mathematics lies in gaining access to new results and relating them to already familiar facts and ideas; here the part played by topology has a special power because of the brevity, flexibility and universality of its language and methods. For instance, the variational principles underlying Morse theory, first seen in relation to the structure of compact surfaces, pervade the theory of manifolds and Lie groups as well as Teichmüller theory and play an essential part in the applications of all this to theoretical physics: the same interaction between topology and optimisation seen in the article by Paolo Teofilatto & Mauro Pontani on orbital trajectories may one day, perhaps by way of string theory, produce a practical application of our developing comprehension of moduli spaces to the real world.

2 Some personal history.

This account of developments in Riemann surfaces drew on my own experience within this field which has fascinated me ever since I first encountered it as a graduate student in 1962. I have tried to make clear what lies behind the mathematics considered in this volume, how such a fusion of topics and combination of methods happened and why it continues to attract wide attention. This falls short of the comprehensive historical treatment the subject merits.

2.1 Mathematics at Birmingham after 1960.

My doctoral adviser A.M. Macbeath, now retired but still active, was unfortunately unable to attend the Anogia conference. Instead he has written the following brief comments on the early days of my postgraduate career.

Bill was a new graduate student (in 1962) when I was a new kid on the block too, coming from Dundee to succeed Peter Hilton (as chairman of the Department of Mathematics) at the University of Birmingham. It was a landmark in my life as well as his. Bill had planned to work with Bill Parry in Ergodic Theory, but Parry took leave that year and Bill began his research with me.

I had the challenge of adapting to the way of life in a new environment. Birmingham was very different from Dundee, where the mathematics department belonged to a college of the venerable University of St. Andrews. At that time my interest had changed from number theory and convex sets to automorphisms of Riemann surfaces. Like Ergodic Theory, this is a branch of Analysis, but our approach, using Fuchsian groups and uniformization, was quite algebraic. With this method Bill solved the problem of finding the lowest genus on which a cyclic group of given order acts. His result had as an immediate corollary the classical result of Wiman giving the maximum order 2(2g+1) for automorphisms of a curve of genus g, but the answer to Bill's question depended on the prime factorisation of the prescribed order and was more intricate. This was included in his doctoral thesis.

I had to cope too with administrative and other matters, so Bill and his fellow graduate students (Hugh Wilkie and Colin Maclachlan arrived at the same time) were often left to their own devices, but they saw problems for themselves, talked to one another as well as to me. They experimented with various types of finite group acting on surfaces of low genus, and several further results soon emerged, for instance with Colin finding the maximum order of a group (rather than an element) acting on a surface of given genus. (The classical Hurwitz maximum 84(g-1) is attained only for a sparsely distributed set of values of g.)

When Bill submitted his work for publication, the manuscript got mislaid somewhere. This is no fun at any stage and can be a devastating matter for a first-time author. After a time we sorted things out and Bill's work was published in the Oxford Quarterly Journal of Mathematics. By this time he had graduated and moved across the Atlantic (in 1966) to work with Lipman Bers. Riemann surfaces with non-trivial automorphisms are the branch points of the Teichmüller orbifold, so Bill's previous work fitted in well.

In New York Bill met Michele Linch, also a disciple of Lipman Bers, and they have been together ever since. They returned to the UK (in 1972) and have both held positions in London since then (Bill at Kings College London and Michele at the LSE). Sometimes on one side of *the Atlantic, sometimes on the other, we have met at conferences and meetings. Bill edited the proceedings of the Cambridge conference in 1976. His direct research contributions include the invention of the (so-called) curve complex.*

Perhaps a few comments (from me) are in order here. Colin and I did indeed start as graduate students together and have remained firm friends to this day. Lacking any real knowledge of German, we painfully translated together Wiman's classic papers of 1892 (on automorphism groups of low genus curves) into a form of quasi-English, and spent much time looking into the Todd-Coxeter enumerative technique for cosets in discrete groups, a topic on which I was not at all keen. Fortunately it was soon possible for me to concentrate on something simpler: finite cyclic groups are a comfort. Colin's result is that the maximal automorphism group in genus g has order at least $8g + 8$; it was obtained by different methods at around the same time by Bob Accola at Brown. Murray was the first European to appreciate the true significance of the Ahlfors-Bers theory; he introduced us to it in a seminar on Ahlfors's work in 1964, and applied Teichmüller's extremal mapping theorem to extend the classic Nielsen theorem in surface topology. In the next few years Murray directed many students; both mathematical and social activity were lively in Birmingham.

The reference to the conference volume which I edited, *Discrete Groups and Automorphic Functions* (Academic Press 1976), prompts me to recall just how thinly documented this field was in those days. The Ahlfors book *Lectures on Quasiconformal Mappings* (recently reissued by the American Math. Society) and the Bers Zürich Notes *Moduli of Riemann surfaces* (ETH, 1964) were very influential and so was Irwin Kra's *Automorphic forms and Kleinian groups* (W.A. Benjamin, 1972), but no comprehensive text appeared on Teichmüller spaces until Fred Gardiner's *Teichmüller Theory and Quadratic Differentials* (John Wiley-Interscience, 1987), followed by Subha Nag's *Complex Analytic Theory of Teichmüller Spaces* (John Wiley, 1988) and Bernard Maskit's treatise *Kleinian Groups* (in Springer's Grundlehren series, 1988). In the interim there was little background reading, apart from that volume and Bill Abikoff's valuable Springer Lecture Notes (*Real analytic theory of Teichmüller spaces*). To fill the demand created by the Thurston revolution we had to wait until the Warwick group under David Epstein and Caroline Series began to fill in the geometric background to Thurston's

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As to the curve complex – there are infinitely many of them in fact, one for each value of the genus $g \geq 2$ – this abstract simplicial structure and the intricate part it plays in low dimensional manifold theory epitomises an aspect of mathematics which has always fascinated me, the interplay between geometrically defined groups (such as reflection groups, braid groups and mapping class groups) and more primitive simplicial objects such as buildings in algebraic topology. It has been in the spotlight recently largely because of the part it plays in the monumental resolution of Thurston's ending lamination conjecture by Yair Minsky, but it occurs in work on abstract presentations and other actions of the mapping class group and appears to carry some kind of special status as a base of operations for geometric actions of this group. It also stands in as a surrogate Tits building to provide a key ingredient in N.V. Ivanov's remarkable proof in the style of Mostow rigidity of Royden's theorem for the Teichmüller modular group.

2.2 The New York research community: seminars and conferences.

An intrinsic part of the explosive growth of research in mathematics and the sciences generally post-Sputnik has been a steady supply of new blood in the field from all parts of the globe, thanks to the growth in funding for graduate study in the USA and Western Europe, and the hub of research activity within the mathematical community has been the system of weekly seminars. The importance of this for the mental well-being of the research community cannot be overstated: in my case, a first postdoctoral position at Columbia University and the Bers weekly seminar provided intellectual mother's milk, an intrinsic part of my life as an immigrant, feeding my educational and social needs, a model of its kind that I appreciated immensely and long sought to duplicate in London without much success.

In addition, the NSF in America and also corresponding Science Research Councils in Britain and Europe, have a tradition of supporting specialised conferences, in recognition of their cost effectiveness in sustaining progress; in addition, the research community supports this itself through the various national societies. Beginning from the early 1960s, there was a pattern of summer conferences featuring Teichmüller theory every two or three years, lasting for a week or more, enabling workers in the field to renew contacts, develop friendships across all frontiers, exchange ideas and establish longer term research collaborations; this has grown in significance over the years along with the research community. The original conference series on Riemann surfaces began with a famous Princeton conference on Analytic Functions in 1957, passing through a period of concentration on Riemann surfaces to the broader agenda now in place, with coordinated organising committees such as the Ahlfors-Bers Colloquium in the USA, the Ibero-American Congress of Geometry in Latin America, symposia at the Warwick Mathematics Institute and the Rolf Nevanlinna Colloquium in Europe. In this way, research in Riemann surfaces and associated theories has thrived, gaining a much more varied program, too broad now for any single meeting to encompass. At the same time, the growth of cooperation between national societies and formation of international groups for regional congresses in America, Europe and Asia has changed the mould of larger meetings.

3 Final comments.

The proliferation of conferences, enhanced by the freedom brought by e-mail and wide access to the Internet, has led to a blossoming of co-operative small group research worldwide, representing a potential for development of the global scientific community that is unprecedented. What we do when we do our research may be attributable to personal curiosity or a thirst for fame, but it is fuelled by an urge to communicate, clarified by all those conversations which go on within each circle of interested friends and then at conferences. Lipman Bers liked to say jokingly that the reason why we spend the time and effort we do on our research is "for the grudging respect of a few close friends"; but he always emphasized the importance of the seminar as an essential part of academic life, especially for those too burdened with other commitments to deliver a talk, and at conferences he was both an enthusiastic audience member and the most inspirational of speakers. The legacy of the Bers seminar lives on in all the research groups around the world run by his students and their students, which continue to nurture young talent and provide a forum for research in complex analysis and geometry.

For my part, nothing in my career has given me more pleasure than to help my students find their feet in this corner of mathematics. It is a further source of satisfaction to see my former students, Christos Kourouniotis, Gabino González and Paolo Teofilatto, take on the mantle of researcher and pass on the flame of mathematical excitement to their own students. Long may it continue.

The organisation of conferences like the one at Anogia is always carried out by a small self-selecting set of members of our community whose work often goes unrecognised beyond those who are direct beneficiaries of their efforts. Their true reward is the continuing health of our field, but for this event we are all very grateful to our organisers, Christos Kourouniotis, on whom all the local arrangements depended, Gabino González-Diez and Fred Gardiner.

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