Some general results in combinatorial enumeration

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Abstract

This survey article is devoted to general results in combinatorial enumeration. The first part surveys results on growth of hereditary properties of combinatorial structures. These include permutations, ordered and unordered graphs and hypergraphs, relational structures, and others. The second part advertises four topics in general enumeration: 1. counting lattice points in lattice polytopes, 2. growth of context-free languages, 3. holonomicity (i.e., P-recursiveness) of numbers of labeled regular graphs and 4. ultimate modular periodicity of numbers of MSOL-definable structures.

1 Introduction

We survey some general results in combinatorial enumeration. A *problem* in enumeration is (associated with) an infinite sequence $P = (S_1, S_2, \dots)$ of finite sets S_i. Its *counting function* f_P is given by $f_P(n) = |S_n|$, the cardinality of the set S_n. We are interested in results of the following kind on *general* classes of problems and their counting functions.

Scheme of general results in combinatorial enumeration. *The counting function f_P of every problem P in the class $\mathcal{C}$ belongs to the class of functions $\mathcal{F}$. Formally, $\{f_P \mid P \in \mathcal{C}\} \subset \mathcal{F}$.*

The larger $\mathcal{C}$ is, and the more specific the functions in $\mathcal{F}$ are, the stronger the result. The present overview is a collection of many examples of this scheme.

One can distinguish general results of two types. In *exact results*, $\mathcal{F}$ is a class of explicitly defined functions, for example polynomials or functions defined by recurrence relations of certain type or functions computable in polynomial time. In *asymptotic results*, $\mathcal{F}$ consists of functions defined by asymptotic equivalences or asymptotic inequalities, for example functions growing at most exponentially or functions asymptotic to $n^{(1-1/k)n+o(n)}$ as $n \to \infty$, with the constant $k \geq 2$ being an integer.

The sets S_n in P usually constitute sections of a fixed infinite set. Generally speaking, we take an infinite universe U of combinatorial structures and introduce problems and classes of problems as subsets of U and families of subsets of U, by means of *size functions* $s:\ U \to \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and/or (mostly binary) relations between structures in U. More specifically, we will mention many results falling within the framework of growth of downsets in partially order sets, or posets.

Downsets in posets of combinatorial structures. We consider a nonstrict partial ordering $(U, \prec)$, where $\prec$ is a containment or a substructure relation on a set U of combinatorial structures, and a size function $s:\ U \to \mathbb{N}_0$. Problems P are *downsets* in $(U, \prec)$, meaning that $P \subset U$ and $A \prec B \in P$ implies $A \in P$, and the counting function of P is

$$f_P(n) = \#\{A \in P \mid s(A) = n\}.$$

(More formally, the problem is the sequence of sections $(P \cap U_1, P \cap U_2, \ldots)$ where $U_n = \{A \in U \mid s(A) = n\}$.) Downsets are exactly the sets of the form

$$\mathrm{Av}(F) := \{A \in U \mid A \not\succ B \text{ for every } B \text{ in } F\},\ \ F \subset U.$$

There is a one-to-one correspondence $P \mapsto F = \min(U \backslash P)$ and $F \mapsto P = \mathrm{Av}(F)$ between the family of downsets P and the family of *antichains* F, which are sets of mutually incomparable structures under $\prec$. We call the antichain $F = \min(U \backslash P)$ corresponding to a downset P the *base of* P.

We illustrate the scheme by three examples, all for downsets in posets.

1.1 Three examples

Example 1. Downsets of partitions. Here U is the family of partitions of $[n] = \{1, 2, \ldots, n\}$ for n ranging in $\mathbb{N}$, so U consists of finite sets $S = \{B_1, B_2, \ldots, B_k\}$ of disjoint and nonempty finite subsets B_i of

$\mathbb{N}$, called *blocks*, whose union $B_1 \cup B_2 \cup \cdots \cup B_k = [n]$ for some n in $\mathbb{N}$. Two natural size functions on U are order and size, where the *order*, $\|S\|$, of S is the cardinality, n, of the underlying set and the *size*, $|S|$, of S is the number, k, of blocks. The formula for the number of partitions of $[n]$ with k blocks

$$S(n,k) := \#\{S \in U \mid \|S\| = n, |S| = k\} = \sum_{i=0}^{k} \frac{(-1)^i (k-i)^n}{i!(k-i)!}$$

is a classical result (see [111]); $S(n,k)$ are called *Stirling numbers*. It is already a simple example of the above scheme but we shall go further.

For fixed k, the function $S(n,k)$ is a linear combination with rational coefficients of the exponentials $1^n, 2^n, \ldots, k^n$. So is the sum $S(n,1) + S(n,2) + \cdots + S(n,k)$ counting partitions with order n and size at most k. We denote the set of such partitions $\{S \in U \mid |S| \le k\}$ as $U_{\le k}$. Consider the poset $(U, \prec)$ with $S \prec T$ meaning that there is an *increasing* injection $f : \bigcup S \to \bigcup T$ such that every two elements x, y in $\bigcup S$ lie in the same block of S if and only if $f(x), f(y)$ lie in the same block of T. In other words, $S \prec T$ means that $\bigcup T$ has a subset X of size $\|S\|$ such that T induces on X a partition order-isomorphic to S. Note that $U_{\le k}$ is a downset in $(U, \prec)$. We know that the counting function of $U_{\le k}$ with respect to order n equals $a_1 1^n + \cdots + a_k k^n$ with a_i in $\mathcal{Q}$. What are the counting functions of other downsets? If the size is bounded, as for $U_{\le k}$, they have similar form as shown in the next theorem, proved by Klazar [77]. It is our first example of an exact general enumerative result.

Theorem 1.1 (Klazar). *If P is a downset in the poset of partitions such that $\max_{S \in P} |S| = k$, then there exist a natural number n_0 and polynomials $p_1(x)$, $p_2(x)$, $\ldots$, $p_k(x)$ with rational coefficients such that for every $n > n_0$,*

$$f_P(n) = \#\{S \in P \mid \|S\| = n\} = p_1(n)1^n + p_2(n)2^n + \cdots + p_k(n)k^n.$$

If $\max_{S \in P} |S| = +\infty$, the situation is much more intricate and we are far from having a complete description but the growths of $f_P(n)$ below 2^{n-1} have been determined (see Theorem 2.17 and the following comments). We briefly mention three subexamples of downsets with unbounded size, none of which has $f_P(n)$ in the form of Theorem 1.1. If P consists of all partitions of $[n]$ into intervals of length at most 2, then $f_P(n) = F_n$, the n^{th} Fibonacci number, and so $f_P(n) = b_1\alpha^n + b_2\beta^n$ where $\alpha = \frac{\sqrt{5}-1}{2}$,

$\beta = \frac{\sqrt{5}+1}{2}$ and $b_1 = \frac{\alpha}{\sqrt{5}}, b_2 = \frac{\beta}{\sqrt{5}}$. If P is given as $P = \mathrm{Av}(\{C\})$ where $C = \{\{1,3\},\{2,4\}\}$ (the partitions in P are so called *noncrossing partition*, see the survey of Simion [106]) then $f_P(n) = \frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number which is asymptotically $cn^{-3/2}4^n$. Finally, if $P = U$, so P consists of all partitions, then $f_P(n) = B_n$, the n^{th} Bell number which grows superexponentially.

Example 2. Hereditary graph properties. Here U is the universe of finite simple graphs $G = ([n], E)$ with vertex sets $[n]$, n ranging over $\mathbb{N}$, and $\prec$ is the induced subgraph relation; $G_1 = ([n_1], E_1) \prec G_2 = ([n_2], E_2)$ means that there is an injection from $[n_1]$ to $[n_2]$ (not necessarily increasing) that sends edges to edges and nonedges to nonedges. The size, $|G|$, of a graph G is the number of vertices. Problems are downsets in $(U, \prec)$ and are called *hereditary graph properties*. The next theorem, proved by Balogh, Bollobás and Weinreich [18], describes counting functions of hereditary graph properties that grow no faster than exponentially.

Theorem 1.2 (Balogh, Bollobás and Weinreich)**.** *If P is a hereditary graph property such that for some constant $c > 1$, $f_P(n) = \#\{G \in P \mid |G| = n\} < c^n$ for every n in $\mathbb{N}$, then there exists a natural numbers k and n_0 and polynomials $p_1(x)$, $p_2(x)$, ..., $p_k(x)$ with rational coefficients such that for every $n > n_0$,*

$$f_P(n) = p_1(n)1^n + p_2(n)2^n + \cdots + p_k(n)k^n.$$

The case of superexponential growth of $f_P(n)$ is discussed below in Theorem 2.11.

In both examples we have the same class of functions $\mathcal{F}$, linear combinations $p_1(n)1^n + p_2(n)2^n + \cdots + p_k(n)k^n$ with $p_i \in \mathcal{Q}[x]$. It would be nice to find a common extension of Theorems 1.1 and 1.2. It would be also of interest to determine if the two classes of functions realizable as counting functions in both theorems coincide and how they differ from $\mathcal{Q}[x, 2^x, 3^x, \ldots]$.

Example 3. Downsets of words. Here U is the set of finite words over a finite alphabet A, so $U = \{u = a_1a_2\ldots a_k \mid a_i \in A\}$. The size, $|u|$, of such a word is its length k. The subword relation (also called the factor relation) $u = a_1a_2\ldots a_k \prec v = b_1b_2\ldots b_l$ means that $b_{i+1} = a_1, b_{i+2} = a_2, \ldots, b_{i+k} = a_k$ for some i. We associate with an infinite word $v = b_1b_2\ldots$ over A the set $P = P_v$ of all its finite subwords, thus $P_v = \{b_rb_{r+1}\ldots b_s \mid 1 \le r \le s\}$. Note that P_v is a downset in

$(U, \prec)$. The next theorem was proved by Morse and Hedlund [92], see also Allouche and Shallit [7, Theorem 10.2.6].

Theorem 1.3 (Morse and Hedlund)**.** *Let P be the set of all finite subwords of an infinite word v over a finite alphabet A. Then $f_P(n) = \#\{u \in P \mid |u| = n\}$ is either larger than n for every n in $\mathbb{N}$ or is eventually constant. In the latter case the word v is eventually periodic.*

The case when P is a general downset in $(U, \prec)$, not necessarily coming from an infinite word (cf. Subsection 2.4), is discussed below in Theorem 2.19.

Examples 1 and 2 are exact results and example 3 combines a tight form of an asymptotic inequality with an exact result. Examples 1 and 2 involve only countably many counting functions $f_P(n)$ and, as follows from the proofs, even only countably many downsets P. In example 3 we have uncountably many distinct counting functions. To see this, take $A = \{0, 1\}$ and consider infinite words v of the form $v = 10^{n_1}10^{n_2}10^{n_3}1\ldots$ where $1 \le n_1 < n_2 < n_2 < \ldots$ is a sequence of integers and $0^m = 00\ldots0$ with m zeros. It follows that for distinct words v the counting functions f_{P_v} are distinct; Proposition 2.1 presents similar arguments in more general settings.

1.2 Content of the overview

The previous three examples illuminated to some extent general enumerative results we are interested in but they are not fully representative because we shall cover a larger area than the growth of downsets. We do not attempt to set forth any more formalized definition of a general enumerative result than the initial scheme but in Subsections 2.4 and 3.4 we will discuss some general approaches of finite model theory based on relational structures. Not every result or problem mentioned here fits naturally the scheme; Proposition 2.1 and Theorem 2.6 are rather results to the effect that $\{f_P \mid P \in \mathcal{C}\}$ is too big to be contained in a small class $\mathcal{F}$. This collection of general enumerative results is naturally limited by the author's research area and his taste but we do hope that it will be of interest to others and that it will inspire a quest for further generalizations, strengthenings, refinements, common links, unifications etc.

For the lack of space, time and expertise we do not mention results on growth in algebraic structures, especially the continent of growth in groups; we refer the reader for information to de la Harpe [52] (and also to Cameron [43]). Also, this is not a survey on the class of problems #P in computational complexity theory (see Papadimitriou [94, Chapter 18]). There are other areas of general enumeration not mentioned properly here, for example 0-1 laws (see Burris [42] and Spencer [109]).

In the next subsection we review some notions and definitions from combinatorial enumeration, in particular we recall the notion of Wilfian formula (polynomial-time counting algorithm). In Section 2 we review results on growth of downsets in posets of combinatorial structures. Subsection 2.1 is devoted to pattern avoiding permutations, Subsections 2.2 and 2.3 to graphs and related structures, and Subsection 2.4 to relational structures. Most of the results in Subsections 2.2 and 2.3 were found by Balogh and Bollobás and their coauthors [11, 13, 12, 15, 14, 16, 17, 18, 19, 20, 21]. We recommend the comprehensive survey of Bollobás [30] on this topic. In Section 3 we advertise four topics in general enumeration together with some related results. 1. The Ehrhart–Macdonald theorem on numbers of lattice points in lattice polytopes. 2. Growth of context-free languages. 3. The theorem of Gessel on numbers of labeled regular graphs. 4. The Specker–Blatter theorem on numbers of MSOL-definable structures.

1.3 Notation and some specific counting functions

As above, we write $\mathbb{N}$ for the set $\{1, 2, 3, \dots\}$, $\mathbb{N}_0$ for $\{0, 1, 2, \dots\}$ and $[n]$ for $\{1, 2, \dots, n\}$. We use $\#X$ and $|X|$ to denote the cardinality of a set. By the phrase "for every n" we mean "for every n in $\mathbb{N}$" and by "for large n" we mean "for every n in $\mathbb{N}$ with possibly finitely many exceptions". Asymptotic relations are always based on $n \to \infty$. The *growth constant* $c = c(P)$ of a problem P is $c = \limsup f_P(n)^{1/n}$; the reciprocal $1/c$ is then the radius of convergence of the power series $\sum_{n\ge 0} f_P(n)x^n$.

We review several counting sequences appearing in the mentioned results. *Fibonacci numbers* $(F_n) = (1, 2, 3, 5, 8, 13, \dots)$ are given by the recurrence $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. They are a particular case $F_n = F_{n,2}$ of the *generalized Fibonacci numbers* $F_{n,k}$, given by the recurrence $F_{n,k} = 0$ for $n < 0$, $F_{0,k} = 1$ and $F_{n,k} = F_{n-1,k} + F_{n-2,k} + \cdots + F_{n-k,k}$ for $n > 0$. Using the notation $[x^n]G(x)$ for the coefficient of x^n in the power series expansion of the expression $G(x)$, we have

$$F_{n,k} = [x^n]\frac{1}{1 - x - x^2 - \cdots - x^k}.$$

Standard methods provide asymptotic relations $F_{n,2} \sim c_2(1.618\dots)^n$, $F_{n,3} \sim c_3(1.839\dots)^n$, $F_{n,4} \sim c_4(1.927\dots)^n$ and generally $F_{n,k} \sim c_k\alpha_k^n$ for constants $c_k > 0$ and $1 < \alpha_k < 2$; $1/\alpha_k$ is the least positive root of the denominator $1-x-x^2-\dots-x^k$ and $\alpha_2, \alpha_3, \dots$ monotonically increase to 2. The unlabeled exponential growth of tournaments (Theorem 2.21) is governed by the *quasi-Fibonacci numbers* F_n^* defined by the recurrence $F_0^* = F_1^* = F_2^* = 1$ and $F_n^* = F_{n-1}^* + F_{n-3}^*$ for $n \geq 3$; so

$$F_n^* = [x^n]\frac{1}{1-x-x^3}$$

and $F_n^* \sim c(1.466\dots)^n$.

We introduced *Stirling numbers* $S(n,k)$ in Example 1. The *Bell numbers* $B_n = \sum_{k=1}^n S(n,k)$ count all partitions of an n-elements set and follow the recurrence $B_0 = 1$ and $B_n = \sum_{k=0}^{n-1}\binom{n-1}{k}B_k$ for $n \geq 1$. Equivalently,

$$B_n = [x^n]\sum_{k=0}^{\infty}\frac{x^k}{(1-x)(1-2x)\dots(1-kx)}.$$

The asymptotic form of the Bell numbers is

$$B_n = n^{n(1-\log\log n/\log n + O(1/\log n))}.$$

The numbers p_n of *integer partitions* of n count the ways to express n as a sum of possibly repeated summands from $\mathbb{N}$, with the order of summands being irrelevant. Equivalently,

$$p_n = [x^n]\prod_{k=1}^{\infty}\frac{1}{1-x^k}.$$

The asymptotic form of p_n is $p_n \sim cn^{-1}\exp(d\sqrt{n})$ for some constants $c, d > 0$. See Andrews [8] for more information on these asymptotics and for recurrences satisfied by p_n.

A sequence $f : \mathbb{N} \to \mathbb{C}$ is a *quasipolynomial* if for every n we have $f(n) = a_k(n)n^k + \dots + a_1(n)n + a_0(n)$ where $a_i : \mathbb{N} \to \mathbb{C}$ are periodic functions. Equivalently,

$$f(n) = [x^n]\frac{p(x)}{(1-x)(1-x^2)\dots(1-x^l)}$$

for some l in $\mathbb{N}$ and a polynomial $p \in \mathbb{C}[x]$. We say that the sequence f is *holonomic* (other terms are P-recursive and D-finite) if it satisfies for every n (equivalently, for large n) a recurrence

$$p_k(n)f(n+k) + p_{k-1}(n)f(n+k-1) + \dots + p_0(n)f(n) = 0$$

with polynomial coefficients $p_i \in \mathbb{C}[x]$, not all zero. Equivalently, the power series $\sum_{n\geq 0} f(n)x^n$ satisfies a linear differential equation with polynomial coefficients. Holonomic sequences generalize sequences satisfying linear recurrences with constant coefficients. The sequences $S(n,k)$, $F_{n,k}$, and F_n^* for each fixed k satisfy a linear recurrence with constant coefficients and are holonomic. The sequences of Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ and of factorial numbers $n!$ are holonomic as well. The sequences B_n and p_n are not holonomic [112]. It is not hard to show that if (a_n) is holonomic and every a_n is in $\mathcal{Q}$, then the polynomials $p_i(x)$ in the recurrence can be taken with integer coefficients. In particular, there are only countably many holonomic rational sequences.

Recall that a power series $F = \sum_{n\geq 0} a_n x^n$ with a_n in $\mathbb{C}$ is *algebraic* if there exists a nonzero polynomial $Q(x,y)$ in $\mathbb{C}[x,y]$ such that $Q(x,F(x)) = 0$. F is *rational* if Q has degree 1 in y, that is, $F(x) = R(x)/S(x)$ for two polynomials in $\mathbb{C}[x]$ where $S(0) \neq 0$. It is well known (Comtet [50], Stanley [112]) that algebraic power series have holonomic coefficients and that the coefficients of rational power series satisfy (for large n) linear recurrence with constant coefficients.

Wilfian formulas. A counting function $f_P(n)$ has a *Wilfian formula* (Wilf [117]) if there exists an algorithm that calculates $f_P(n)$ for every input n effectively, that is to say, in polynomial time. More precisely, we require (extending the definition in [117]) that the algorithm calculates $f_P(n)$ in the number of steps polynomial in the quantity

$$t = \max(\log n, \log f_P(n)).$$

This is (roughly) the minimum time needed for reading the input and writing down the answer. In the most common situations when $\exp(n^c) < f_P(n) < \exp(n^d)$ for large n and some constants $d > c > 0$, this amounts to requiring a number of steps polynomial in n. But if $f_P(n)$ is small (say $\log n$) or big (say doubly exponential in n), then one has to work with t in place of n. The class of counting functions with Wilfian formulas includes holonomic sequences but is much more comprehensive than that.

2 Growth of downsets of combinatorial structures

We survey results in the already introduced setting of downsets in posets of combinatorial structures $(U, \prec)$. The function $f_P(n)$ counts structures of size n in the downset P and P can also be defined in terms of forbidden substructures as $P = \mathrm{Av}(F)$. Besides the containment relation

$\prec$ we employ also isomorphism equivalence relation $\sim$ on U and will count *unlabeled* (i.e., nonisomorphic) structures in P. We denote the corresponding counting function $g_P(n)$, so

$$g_P(n) = \#(\{A \in P \mid s(A) = n\}/\sim)$$

is the number of isomorphism classes of structures with size n in P.

Restrictions on $f_P(n)$ and $g_P(n)$ defining the classes of functions $\mathcal{F}$ often have the form of *jumps in growth.* A jump is a region of growth prohibited for counting functions—every counting function resides either below it or above it. There are many kinds of jumps but the most spectacular is perhaps the *polynomial–exponential jump* from polynomial to exponential growth, which prohibits counting functions satisfying $n^k < f_P(n) < c^n$ for large n for any constants $k > 0$ and $c > 1$. For groups, Grigorchuk constructed a finitely generated group having such intermediate growth (Grigorchuk [70], Grigorchuk and Pak [69], [52]), which excludes the polynomial–exponential jump for general finitely generated groups, but a conjecture says that this jump occurs for every finitely presented group. We have seen this jump in Theorems 1.1 and 1.2 (from polynomial growth to growth at least 2^n) and will meet new examples in Theorems 2.4, 2.17, 2.18, 2.21, and 3.3.

If $(U, \prec)$ has an infinite antichain A, then under natural conditions we get uncountably many functions $f_P(n)$. This was observed several times in the context of permutation containment and for completeness we give the argument here again. These natural conditions, which will always be satisfied in our examples, are *finiteness*, for every n there are finitely many structures with size n in U, and *monotonicity*, $s(G) \geq s(H)$ & $G \prec H$ implies $G = H$ for every G, H in U. (Recall that $G \prec G$ for every G.)

Proposition 2.1. *If $(U, \prec)$ and the size function $s(\cdot)$ satisfy the monotonicity and finiteness conditions and $(U, \prec)$ has an infinite antichain A, then the set of counting functions $f_P(n)$ is uncountable.*

Proof. By the assumption on U we can assume that the members of A have distinct sizes. We show that all the counting functions $f_{\mathrm{Av}(F)}$ for $F \subset A$ are distinct and so this set of functions is uncountable. We write simply f_F instead of $f_{\mathrm{Av}(F)}$. If X, Y are two distinct subsets of A, we express them as $X = T \cup \{G\} \cup U$ and $Y = T \cup \{H\} \cup V$ so that, without loss of generality, $m = s(G) < s(H)$, and $G_1 \in T, G_2 \in U$ implies $s(G_1) < s(G) < s(G_2)$ and similarly for Y (the sets T, U, V may be empty). Then, by the assumption on $\prec$ and $s(\cdot)$,

$$f_X(m) = f_{T\cup\{G\}}(m) = f_T(m) - 1 = f_{T\cup\{H\}\cup V}(m) - 1 = f_Y(m) - 1$$

and $f_X \neq f_Y$. □

An infinite antichain thus gives not only uncountably many downsets but in fact uncountably many counting functions. Then, in particular, almost all counting functions are not computable because we have only countably many algorithms. Recently, Albert and Linton [4] significantly refined this argument by showing how certain infinite antichains of permutations produce even uncountably many growth constants, see Theorem 2.6.

On the other hand, if every antichain is finite then there are only countably many functions $f_P(n)$. Posets with no infinite antichain are called *well quasiorderings* or shortly *wqo*. (The second part of the wqo property, nonexistence of infinite strictly descending chains, is satisfied automatically by the monotonicity condition.) But even if $(U, \prec)$ has infinite antichains, there still may be only countably many downsets P with slow growth functions $f_P(n)$. For example, this is the case in Theorems 1.1 and 1.2. It is then of interest to determine for which growth uncountably many downsets appear (cf. Theorem 2.5). The posets $(U, \prec)$ considered here usually have infinite antichains, with two notable wqo exceptions consisting of the minor ordering on graphs and the subsequence ordering on words over a finite alphabet.

2.1 Permutations

Let U denote the universe of permutations represented by finite sequences $b_1 b_2 \dots b_n$ such that $\{b_1, b_2, \dots, b_n\} = [n]$. The size of a permutation $\pi = a_1 a_2 \dots a_m$ is its length $|\pi| = m$. The containment relation on U is defined by $\pi = a_1 a_2 \dots a_m \prec \rho = b_1 b_2 \dots b_n$ if and only if for some increasing injection $f : [m] \to [n]$ one has $a_r < a_s \iff b_{f(r)} < b_{f(s)}$ for every r, s in $[m]$. Problems P are downsets in $(U, \prec)$ and their counting functions are $f_P(n) = \#\{\pi \in P \mid |\pi| = n\}$. The poset of permutations $(U, \prec)$ has infinite antichains (see Aktinson, Murphy, and Ruškuc [10]). For further information and background on the enumeration of downsets of permutations see Bóna [34].

Recall that $c(P) = \limsup f_P(n)^{1/n}$. We define

$$E = \{c(P) \in [0, +\infty] \mid P \text{ is a downset of permutations}\}$$