I

Jordan Domains

To begin we construct harmonic measure and solve the Dirichlet problem in the upper half-plane and the unit disc. We next prove the Fatou theorem on nontangential limits. Then we construct harmonic measure on domains bounded by Jordan curves, via the Riemann mapping theorem and the Carathéodory theorem on boundary correspondence. We review two topics from classical complex analysis, the hyperbolic metric and the elementary distortion theory for univalent functions. We conclude the chapter with the theorem of Hayman and Wu on lengths of level sets. Its proof is an elementary application of harmonic measure and the hyperbolic metric.

1. The Half-Plane and the Disc

Write $\mathbb{H} = \{z : \text{Im} z > 0\}$ for the upper half-plane and $\mathbb{R}$ for the real line. Suppose $a < b$ are real. Then the function

$$\theta = \theta(z) = \arg\left(\frac{z-b}{z-a}\right) = \text{Im}\log\left(\frac{z-b}{z-a}\right)$$

is harmonic on $\mathbb{H}$, and $\theta = \pi$ on (a, b) and $\theta = 0$ on $\mathbb{R} \setminus [a, b]$.

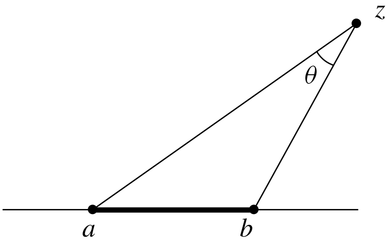

Figure I.1 The harmonic function $\theta(z)$.

Viewed geometrically, $\theta(z) = \mathrm{Re}\varphi(z)$ where $\varphi(z)$ is any conformal mapping from $\mathbb{H}$ to the strip $\{0 < \mathrm{Re}z < \pi\}$ which maps (a, b) onto $\{\mathrm{Re}z = \pi\}$ and $\mathbb{R} \setminus [a, b]$ into $\{\mathrm{Re}z = 0\}$. Let $E \subset \mathbb{R}$ be a finite union of open intervals and write $E = \bigcup_{j=1}^{n}(a_j, b_j)$ with $b_{j-1} < a_j < b_j$. Set

$$\theta_j = \theta_j(z) = \arg\left(\frac{z - b_j}{z - a_j}\right)$$

and define the **harmonic measure** of E at $z \in \mathbb{H}$ to be

$$\omega(z, E, \mathbb{H}) = \sum_{j=1}^{n} \frac{\theta_j}{\pi}. \tag{1.1}$$

Then

(i) $0 < \omega(z, E, \mathbb{H}) < 1$ for $z \in \mathbb{H}$,
(ii) $\omega(z, E, \mathbb{H}) \to 1$ as $z \to E$, and
(iii) $\omega(z, E, \mathbb{H}) \to 0$ as $z \to \mathbb{R} \setminus \overline{E}$.

The function $\omega(z, E, \mathbb{H})$ is the unique harmonic function on $\mathbb{H}$ that satisfies (i), (ii), and (iii). The uniqueness of $\omega(z, E, \mathbb{H})$ is a consequence of the following lemma, known as **Lindelöf's maximum principle**.

Lemma 1.1 (Lindelöf). *Suppose the function $u(z)$ is harmonic and bounded above on a region Ω such that $\overline{\Omega} \neq \mathbb{C}$. Let F be a finite subset of $\partial\Omega$ and suppose*

$$\limsup_{z\to\zeta} u(z) \leq 0 \tag{1.2}$$

for all $\zeta \in \partial\Omega \setminus F$. Then $u(z) \leq 0$ on Ω.

Proof. Fix $z_0 \notin \overline{\Omega}$. Then the map $1/(z - z_0)$ transforms Ω into a bounded region, and thus we may assume Ω is bounded. If (1.2) holds for all $\zeta \in \partial\Omega$, then the lemma is the ordinary maximum principle. Write $F = \{\zeta_1, \ldots, \zeta_N\}$, let $\varepsilon > 0$, and set

$$u_\varepsilon(z) = u(z) - \varepsilon \sum_{j=1}^{N} \log\left(\frac{\mathrm{diam}(\Omega)}{|z - \zeta_j|}\right).$$

Then u_ε is harmonic on Ω and $\limsup_{z\to\zeta} u_\varepsilon(z) \leq 0$ for all $\zeta \in \partial\Omega$. Therefore $u_\varepsilon \leq 0$ for all ε, and

$$u(z) \leq \lim_{\varepsilon\to 0} \varepsilon \sum_{j=1}^{N} \log\left(\frac{\mathrm{diam}(\Omega)}{|z - \zeta_j|}\right) = 0. \qquad \blacksquare$$

Lindelöf [1915] proved Lemma 1.1 under the weaker hypothesis that $\partial\Omega$ is infinite. See also Ahlfors [1973]. Exercise 3 and Exercise II.3 tell more about Lindelöf's maximum principle.

Given a domain Ω and a function $f \in C(\partial\Omega)$, the **Dirichlet problem** for f on Ω is to find a function $u \in C(\overline{\Omega})$ such that $\Delta u = 0$ on Ω and $u|_{\partial\Omega} = f$. Theorem 1.2 treats the Dirichlet problem on the upper half-plane $\mathbb{H}$.

Theorem 1.2. *Suppose $f \in C(\mathbb{R} \cup \{\infty\})$. Then there exists a unique function $u = u_f \in C(\overline{\mathbb{H} \cup \{\infty\}})$ such that u is harmonic on $\mathbb{H}$ and $u|_{\partial\mathbb{H}} = f$.*

Proof. We can assume f is real valued and $f(\infty) = 0$. For $\varepsilon > 0$, take disjoint open intervals $I_j = (t_j, t_{j+1})$ and real constants c_j, $j = 1, \dots, n$, so that the step function

$$f_\varepsilon(t) = \sum_{j=1}^{n} c_j \chi_{I_j}$$

satisfies

$$\left\| f_\varepsilon - f \right\|_{L^\infty(\mathbb{R})} < \varepsilon. \tag{1.3}$$

Set

$$u_\varepsilon(z) = \sum_{j=1}^{n} c_j \omega(z, I_j, \mathbb{H}).$$

If $t \in \mathbb{R} \setminus \bigcup \partial I_j$, then

$$\lim_{\mathbb{H} \ni z \to t} u_\varepsilon(z) = f_\varepsilon(t)$$

by (ii) and (iii). Therefore by (1.3) and Lemma 1.1,

$$\sup_{\mathbb{H}} \left| u_{\varepsilon_1}(z) - u_{\varepsilon_2}(z) \right| < \varepsilon_1 + \varepsilon_2.$$

Consequently the limit

$$u(z) \equiv \lim_{\varepsilon \to 0} u_\varepsilon(z)$$

exists, and the limit $u(z)$ is harmonic on $\mathbb{H}$ and satisfies

$$\sup_{\mathbb{H}} |u(z) - u_\varepsilon(z)| \le 2\varepsilon.$$

We claim that

$$\limsup_{z \to t} |u_\varepsilon(z) - f(t)| \le \varepsilon \tag{1.4}$$

for all $t \in \mathbb{R}$. It is clear that (1.4) holds when $t \notin \bigcup \partial I_j$. To verify (1.4) at the endpoint $t_{j+1} \in \partial I_j \cap \partial I_{j+1}$, notice that by (ii), (iii), and Lemma 1.1,

$$\sup_{\mathbb{H}}\left|c_j\omega(z, I_j, \mathbb{H}) + c_{j+1}\omega(z, I_{j+1}, \mathbb{H}) - \left(\frac{c_j + c_{j+1}}{2}\right)\omega(z, I_j \cup I_{j+1}, \mathbb{H})\right| \leq \left|\frac{c_j - c_{j+1}}{2}\right|,$$

while

$$\lim_{z\to t_{j+1}}\left(\frac{c_j + c_{j+1}}{2}\right)\omega(z, I_j \cup I_{j+1}, \mathbb{H}) = \frac{c_j + c_{j+1}}{2}.$$

Hence all limit values of $u_\varepsilon(z)$ at t_{j+1} lie in the closed interval with endpoints c_j and c_{j+1}, and then (1.3) yields (1.4) for the endpoint t_{j+1}.

Now let $t \in \mathbb{R}$. By (1.4)

$$\limsup_{z\to t}|u(z) - f(t)| \leq \sup_{z\in\mathbb{H}}|u(z) - u_\varepsilon(z)| + \limsup_{z\to t}|u_\varepsilon(z) - f(t)| \leq 3\varepsilon.$$

The same estimate holds if $t = \infty$. Therefore u extends to be continuous on $\overline{\mathbb{H}}$ and $u|_{\partial\mathbb{H}} = f$. The uniqueness of u follows immediately from the maximum principle. ■

For $a < b$, elementary calculus gives

$$\omega(x + iy, (a, b), \mathbb{H}) = \frac{1}{\pi}\left(\tan^{-1}\left(\frac{x-a}{y}\right) - \tan^{-1}\left(\frac{x-b}{y}\right)\right) = \int_a^b \frac{y}{(t-x)^2 + y^2}\frac{dt}{\pi}.$$

If $E \subset \mathbb{R}$ is measurable, we define the **harmonic measure** of E at $z \in \mathbb{H}$ to be

$$\omega(z, E, \mathbb{H}) = \int_E \frac{y}{(t-x)^2 + y^2}\frac{dt}{\pi}. \tag{1.5}$$

When E is a finite union of open intervals this definition (1.5) is the same as definition (1.1). For $z = x + iy \in \mathbb{H}$, the density

$$P_z(t) = \frac{1}{\pi}\frac{y}{(x-t)^2 + y^2}$$

is called the **Poisson kernel** for $\mathbb{H}$. If $f \in C(\mathbb{R} \cup \{\infty\})$, the proof of Theorem 1.2 shows that

$$u_f(z) = \int_{\mathbb{R}} f(t)P_z(t)dt,$$

and for this reason u_f is also called the **Poisson integral** of f.

Note that the harmonic measure $\omega(z, E, \Omega)$ is a harmonic function in its first variable z and a probability measure in its second variable E. If $z_1, z_2 \in \mathbb{H}$ then

$$0 < C^{-1} \leq \frac{\omega(z_1, E, \mathbb{H})}{\omega(z_2, E, \mathbb{H})} \leq C < \infty,$$

where C depends on z_1 and z_2 but not on E. This inequality, known as **Harnack's inequality**, is easily proved by comparing the kernels in (1.5).

Now let $\mathbb{D}$ be the unit disc $\{z : |z| < 1\}$ and let E be a finite union of open arcs on $\partial\mathbb{D}$. Then we define the **harmonic measure** of E at z in $\mathbb{D}$ to be

$$\omega(z, E, \mathbb{D}) \equiv \omega(\varphi(z), \varphi(E), \mathbb{H}), \tag{1.6}$$

where φ is any conformal map of $\mathbb{D}$ onto $\mathbb{H}$. This harmonic function satisfies conditions analogous to (i), (ii), and (iii), so that by Lemma 1.1 the definition (1.6) does not depend on the choice of φ. It follows by the change of variables $\varphi(z) = i(1+z)/(1-z)$ that

$$\omega(z, E, \mathbb{D}) = \int_E \frac{1-|z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}.$$

An equivalent way to find this function is by a construction similar to (1.1). This construction is outlined in Exercise 1.

Theorem 1.3. *Let $f(e^{i\theta})$ be an integrable function on $\partial\mathbb{D}$ and set*

$$u(z) = u_f(z) = \int_0^{2\pi} f(e^{i\theta}) \frac{1-|z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}. \tag{1.7}$$

Then $u(z)$ is harmonic on $\mathbb{D}$. If f is continuous at $e^{i\theta_0} \in \partial\mathbb{D}$, then

$$\lim_{\mathbb{D} \ni z \to e^{i\theta_0}} u(z) = f(e^{i\theta_0}). \tag{1.8}$$

Clearly (1.8) also holds if the integrable function f is changed on a measure zero subset of $\partial\mathbb{D} \setminus \{e^{i\theta_0}\}$. The function $u = u_f$ is called the **Poisson integral** of f and the kernel

$$P_z(\theta) = \frac{1}{2\pi} \frac{1-|z|^2}{|e^{i\theta} - z|^2}$$

is the **Poisson kernel** for the disc. If $f \in C(\partial\mathbb{D})$ then

$$U(z) = \begin{cases} u_f(z), & z \in \mathbb{D} \\ f(z), & z \in \partial\mathbb{D} \end{cases}$$

is the solution of the **Dirichlet problem** for f on $\mathbb{D}$.

In the special case when $f(e^{i\theta})$ is continuous, Theorem 1.3 follows from Theorem 1.2 and a change of variables. Conversely, Theorem 1.3 shows that

Theorem 1.2 can be extended to $f \in L^1(dt/(1+t^2))$, again by changing variables.

Proof of Theorem 1.3. We may suppose f is real valued. From the identity

$$\mathrm{Re}\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big) = 2\pi P_z(\theta),$$

we see that u is the real part of the analytic function

$$\int_0^{2\pi} f(e^{i\theta})\frac{e^{i\theta}+z}{e^{i\theta}-z}\frac{d\theta}{2\pi},$$

and therefore that u is a harmonic function. One can also see u is harmonic by differentiating the integral (1.7).

Suppose f is continuous at $e^{i\theta_0}$ and let $\varepsilon > 0$. Then

$$|f(e^{i\theta}) - f(e^{i\theta_0})| < \varepsilon$$

on an interval $I = (\theta_1, \theta_2)$ containing θ_0. Setting

$$u_\varepsilon(z) = \int_{[0,2\pi]\setminus I} \frac{1-|z|^2}{|e^{i\theta}-z|^2} f(e^{i\theta})\frac{d\theta}{2\pi} + f(e^{i\theta_0})\omega(z, I, \mathbb{D}),$$

we have

$$|u(z) - u_\varepsilon(z)| = \left|\int_I \frac{1-|z|^2}{|e^{i\theta}-z|^2}(f(e^{i\theta}) - f(e^{i\theta_0}))\frac{d\theta}{2\pi}\right| \le \varepsilon\omega(z, I, \mathbb{D}) \le \varepsilon.$$

However, $\lim_{z\to e^{i\theta_0}} u_\varepsilon(z) = f(e^{i\theta_0})$ by the definition of u_ε. Therefore

$$\limsup_{z\to e^{i\theta_0}} |u(z) - f(e^{i\theta_0})| < \varepsilon,$$

and (1.8) holds when f is continuous at $e^{i\theta_0}$. ■

2. Fatou's Theorem and Maximal Functions

When $f \in L^1(\partial\mathbb{D})$ the limit (1.8) can fail to exist at every $\zeta \in \partial\mathbb{D}$; see Exercise 7. However, there is a substitute result known as **Fatou's theorem**, in which the approach $z \to \zeta$ is restricted to cones. For $\zeta \in \partial\mathbb{D}$ and $\alpha > 1$, we define the **cone**

$$\Gamma_\alpha(\zeta) = \big\{z : |z-\zeta| < \alpha(1-|z|)\big\}.$$

The cone $\Gamma_\alpha(\zeta)$ is asymptotic to a sector with vertex ζ and angle $2\sec^{-1}(\alpha)$ that is symmetric about the radius $[0, \zeta]$. The cones $\Gamma_\alpha(\zeta)$ expand as α increases.

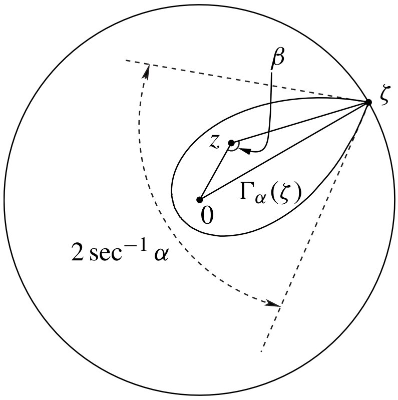

Figure I.2 The cone $\Gamma_\alpha(\zeta)$.

A function $u(z)$ on $\mathbb{D}$ has **nontangential limit** A at $\zeta \in \partial\mathbb{D}$ if

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} u(z) = A \tag{2.1}$$

for *every* $\alpha > 1$. A good example is the function $u(z) = e^{\frac{z+1}{z-1}}$. This function $u(z)$ is continuous on $\partial\mathbb{D} \setminus \{1\}$, and $|u(\zeta)| = 1$ on $\partial\mathbb{D} \setminus \{1\}$, but $u(z)$ has nontangential limit 0 at $\zeta = 1$. With fixed $\alpha > 1$, the **nontangential maximal function** of u at ζ is

$$u^*_\alpha(\zeta) = \sup_{\Gamma_\alpha(\zeta)} |u(z)|.$$

If u has a finite nontangential limit at ζ, then $u^*_\alpha(\zeta) < \infty$ for every $\alpha > 1$.

We write $|E|$ for the Lebesgue measure of $E \subset \partial\mathbb{D}$.

Theorem 2.1 (Fatou's theorem). *Let $f(e^{i\theta}) \in L^1(\partial\mathbb{D})$ and let $u(z)$ be the Poisson integral of f. Then at almost every $\zeta = e^{i\theta} \in \partial\mathbb{D}$,*

$$\lim_{\Gamma_\alpha(\zeta)\ni z\to\zeta} u(z) = f(\zeta) \tag{2.2}$$

for all $\alpha > 1$. Moreover, for each $\alpha > 1$

$$\Big|\{\zeta \in \partial\mathbb{D} : u^*_\alpha(\zeta) > \lambda\}\Big| \le \frac{3+6\alpha}{\lambda}||f||_1. \tag{2.3}$$

When $u(z)$ is the Poisson integral of $f \in L^1(\partial\mathbb{D})$ the function $u = u_f$ is also called the **solution to the Dirichlet problem for** f, even though u converges to f on $\partial\mathbb{D}$ only nontangentially and only almost everywhere.

Inequality (2.3) says the operator $L^1(\partial\mathbb{D}) \ni f \to u^*_\alpha$ is **weak-type 1-1**. It follows from (2.2) that $u^*_\alpha(\zeta) < \infty$ almost everywhere, but (2.3) is a sharper, quantitative result. In the proof of the theorem we derive (2.2) from the estimate (2.3).

The proof of Fatou's theorem is a standard approximate identity argument from real analysis that derives almost everywhere convergence for all $f \in L^1(\partial\mathbb{D})$ from

(a) an estimate such as (2.3) for the maximal function, and

(b) the almost everywhere convergence (2.2) for all functions in a dense subset of $L^1(\partial\mathbb{D})$, such as $C(\partial\mathbb{D})$.

See Stein [1970]. We will use this approximate identity argument again later.

Proof. As promised, we first assume (2.3) and show (2.3) implies (2.2). Fix α temporarily. We may assume f is real valued. Set

$$W_f(\zeta) = \limsup_{\Gamma_\alpha \ni z \to \zeta} |u_f(z) - f(\zeta)|.$$

Then $W_f(\zeta) \le u_\alpha^*(\zeta) + |f(\zeta)|$. Chebyshev's inequality gives

$$\big|\{\zeta : |f(\zeta)| > \lambda\}\big| \le \frac{||f||_1}{\lambda},$$

so that by (2.3),

$$\begin{aligned} \big|\{\zeta : W_f(\zeta) > \lambda\}\big| &\le \big|\{\zeta : u_\alpha^*(\zeta) > \lambda/2\}\big| + \big|\{\zeta : |f(\zeta)| > \lambda/2\}\big| \\ &\le \frac{8 + 12\alpha}{\lambda} ||f||_1. \end{aligned} \tag{2.4}$$

Fix $\varepsilon > 0$ and let $g \in C(\partial\mathbb{D})$ be such that $||f - g||_1 \le \varepsilon^2$. Now $W_g(\zeta) = 0$ by Theorem 1.3, and hence

$$W_f(\zeta) = W_{f-g}(\zeta).$$

Applying (2.4) to $f - g$ then gives

$$\big|\{\zeta : W_f(\zeta) > \varepsilon\}\big| \le \frac{(8 + 12\alpha)\varepsilon^2}{\varepsilon} = (8 + 12\alpha)\varepsilon.$$

Therefore, for any fixed α, (2.2) holds almost everywhere. Because the cones Γ_α increase with α, it follows that (2.2) holds for every $\alpha > 1$, except for ζ in a set of measure zero.

To prove (2.3) we will dominate the nontangential maximal function with a second, simpler maximal function. Let $f \in L^1(\partial\mathbb{D})$ and write

$$Mf(\zeta) = \sup_{I \ni \zeta} \frac{1}{|I|} \int_I |f| d\theta$$

for the maximal average of $|f|$ over subarcs $I \subset \partial\mathbb{D}$ that contain ζ. The function Mf is called the **Hardy–Littlewood maximal function** of f. The function Mf is simpler than u_α^* because it features characteristic functions of intervals instead of Poisson kernels.

Lemma 2.2. *Let $u(z)$ be the Poisson integral of $f \in L^1(\partial\mathbb{D})$ and let $\alpha > 1$. Then*

$$u_\alpha^*(\zeta) \le (1 + 2\alpha)Mf(\zeta). \tag{2.5}$$

Proof. Assume $\zeta = 1$. Fix z so that $\theta_0 = \arg z$ has $|\theta_0| \le \pi$. Define

$$\begin{aligned} P_z^*(\theta) &= \sup\{P_z(\varphi) : |\theta| \le |\varphi| \le \pi\} \\ &= \begin{cases} \dfrac{1}{2\pi}\dfrac{1+|z|}{1-|z|}, & |\theta| \le |\theta_0| \\ \max(P_z(\theta), P_z(-\theta)), & |\theta_0| < |\theta| \le \pi. \end{cases} \end{aligned}$$

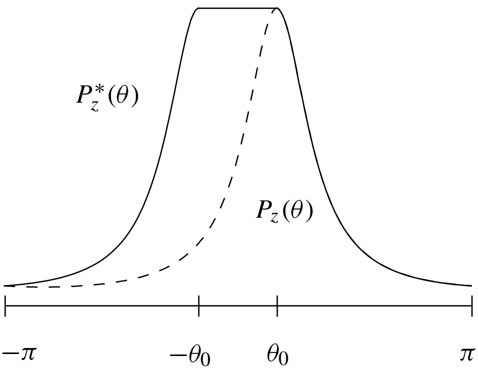

Figure I.3 The function P_z^*.

The function P_z^* satisfies

(i) $P_z^*(\theta)$ is an even function of $\theta \in [-\pi, \pi]$,
(ii) $P_z^*(\theta)$ is decreasing on $[0, \pi]$, and
(iii) $P_z^*(\theta) \ge P_z(\theta)$.

The even function P_z^* is the smallest decreasing majorant of P_z on $[0, \pi]$. We may assume $f(e^{i\theta}) \ge 0$, so that

$$\int f(e^{i\theta})P_z(\theta)d\theta \le \int f(e^{i\theta})P_z^*(\theta)d\theta.$$

Then properties (i) and (ii) imply

$$\int f(e^{i\theta})P_z^*(\theta)d\theta \leq ||P_z^*||_1 Mf(1) \tag{2.6}$$

because P_z^* is the increasing limit of a sequence of functions of the form

$$\sum c_j \left(\frac{1}{2\theta_j}\chi_{(-\theta_j,\theta_j)}(\theta)\right)$$

with $c_j \geq 0$ and $\sum c_j \leq ||P_z^*||_1$.

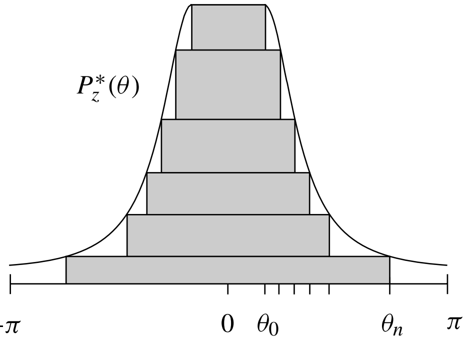

Figure I.4 Approximating P_z^* by a step function.

Now we claim that when $z \in \Gamma_\alpha(1)$,

$$||P_z^*||_1 \leq (1 + 2\alpha). \tag{2.7}$$

Note that (iii), (2.6), and (2.7) imply (2.5). To prove (2.7) we first assume $-\pi/2 \leq \theta_0 = \arg z \leq \pi/2$. Then by the law of sines,

$$\frac{|\theta_0|}{1-|z|} \leq \alpha \frac{|\theta_0|}{|1-z|} \leq \frac{\pi\alpha}{2}\frac{|\sin\theta_0|}{|1-z|} = \frac{\pi\alpha}{2}\frac{|\sin\beta|}{1} \leq \frac{\pi\alpha}{2},$$

where $\beta = \arg(z-1)/z$ is explained by Figure I.2. If $\pi/2 \leq |\theta_0| \leq \pi$ and $z \in \Gamma_\alpha(1)$, then $|1-z| \geq 1$ and

$$\frac{|\theta_0|}{1-|z|} \leq \alpha \frac{|\theta_0|}{|1-z|} \leq \pi\alpha.$$

Hence (see Figure I.3)

$$\|P_z^*\|_1 = 2\int_{|\theta_0|}^{\pi} P_z(\theta)d\theta + \frac{2|\theta_0|}{2\pi}\frac{1+|z|}{1-|z|} \leq (1+2\alpha).$$

That proves (2.7) and therefore Lemma 2.2. ∎