A COURSE
OF
PURE MATHEMATICS

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BY

G. H. HARDY

TENTH EDITION

WITH A FOREWORD
BY T. W. KÖRNER

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FOREWORD

T. W. Körner

My copy of Hardy's *Pure Mathematics* is the eighth edition, printed in 1941. It must have been one of the first books that my father bought as an almost penniless refugee student in England, and the pencilled notations show that he read most of it. It was the first real mathematics book that I attempted to read and, though much must have passed over my head, I can still feel the thrill of reading the construction of the real numbers by Dedekind cuts. One hundred years after it was first published, CUP is issuing this Centenary edition, not as an act of piety, but because *A Course In Pure Mathematics* remains an excellent seller, bought and read by every new generation of mathematicians.

During most of the nineteenth century, mathematics stood supreme among the subjects studied at Cambridge. Exposure to the absolute truths of mathematics was an essential part of an intellectual education. The most able students could measure themselves against their opponents in mathematical examinations (the Tripos) which tested speed, accuracy and problem-solving abilities to the utmost. However, it was a system directed entirely towards the teaching of undergraduates. In Germany and France there were research schools in centres like Berlin, Göttingen and Paris. In England, major mathematicians like Henry Smith and Cayley remained admired but isolated.

An education that produced Maxwell, Kelvin, Rayleigh and Stokes cannot be dismissed out of hand, but any mathematical school which concentrates on teaching and examining runs the risk of becoming old-fashioned. (Think of the concours for the Grandes Écoles in our day.) It is possible that, even in applied mathematics, the Cambridge approach was falling behind Europe. It is certain that, with a few notable but isolated exceptions, pure mathematical research hardly existed in Britain. Hardy took pleasure in repeating the judgement of an unnamed European colleague that the characteristics of English mathematics had been 'occasional flashes of insight, isolated achievements sufficient to show that the ability is really there, but for the most part, amateurism, ignorance, incompetence and triviality.'

When Hardy arrived as a student at Cambridge, reform was very much in the air.

> I had of course found at school, as every future mathematician does, that I could often do things much better than my teachers; and even at Cambridge I found, though naturally much less frequently, that I could sometimes do things better than the college lecturers. But I was really quite ignorant, even when I took the Tripos, of the subjects on which I have spent my life; and I still thought of mathematics as essentially a 'competitive' subject. My eyes were first opened by Professor Love, who taught me for a few terms and gave me my first serious conception of analysis. But the great debt which I owe to him – he was after all primarily an applied mathematician – was his advice to read Jordan's famous *Cours d'analyse*; and I shall never forget the astonishment with which I read this remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant.
>
> [*A Mathematician's Apology*]

Ever since Newton, mathematicians had struggled with the problem of putting the calculus on as sound a footing as Euclid's geometry. But what were the fundamental axioms on which the calculus was to be founded? How should concepts like a differentiable function be defined? Which theorems were 'obvious' and which 'subtle'? Until these questions were answered, all calculus textbooks would have to mix accurate argument with hand waving. Sometimes the author would be aware of gaps and resort to rhetoric 'Persist and faith will come to you.' More frequently, author and reader would sleepwalk hand in hand through the difficulty – most lecturers will be aware how fatally easy it is to convince an audience of an erroneous proof provided you are convinced of it yourself.

The first edition of Jordan's work (1882–87) belonged to this old tradition but the second edition (1893–96) wove together the work of rigorisers like Weierstrass to produce a complete and satisfactory account of the calculus. The impact on Cambridge of Jordan and of the new 'continental' analysis was immense. Young and Hobson, men who had expected to spend their lifetimes in the comfortable routine of undergraduate teaching, suddenly threw themselves into research and, still more remarkably, became great mathematicians.

This impact can be read in three books which are still in print today. We read them now in various revised editions, but all were first written by young men determined to challenge a century of tradition. The first was Whittaker's *A Course of Modern Analysis* (1902) (later editions by Whittaker and Watson), which showed that the special functions which formed the crown jewels of the old analysis were best treated by modern methods. The second was Hobson's *The Theory of Functions* *of a real variable* (1907), which set out the new analysis for professional mathematicians. The third is the present text, first published in 1908 and intended for 'first year students . . . whose abilities reach or approach . . . scholarship standard'.

The idea of such a text may appear equally absurd to those who deal with the mass university system of today and to those whose view of the old university system is moulded by *Brideshead Revisited* or *Sinister Street*. However, although most of the students in Cambridge came from well-off backgrounds, some came from poorer backgrounds and *needed* to distinguish themselves whilst some of their richer companions *wished* to distinguish themselves. Most of the mathematically able students came from a limited number of schools where they often received an outstanding mathematical education. (Read, for example, Littlewood's account of his mathematical education in *A Mathematician's Miscellany*.)

Hardy's intended audience was small and it is not surprising that CUP made him pay £15 out of his own pocket for corrections. This audience was, however, an audience fully accustomed through the study of Euclidean geometry both to follow and, even more importantly, to construct long chains of reasoning. It was also trained in fast and accurate manipulations, both in algebra and calculus, within a problem-solving context. The modern writer of a first course in analysis must address an audience with much less experience of proof, substantially lower algebraic fluency and little experience of applying calculus to interesting problems in mechanics and geometry. Spivak's *Calculus* is outstanding, but Hardy can illustrate his text with much richer exercises. (The reader should note that questions like Example 1.1, which appear to be simple statements are, in fact, invitations to prove those statements.)

Cambridge and Oxford used Hardy's *Pure Mathematics*, and the two universities dominated the British mathematical scene. (Before World War II, almost every mathematics professor in the British Isles was Cambridge or Oxford trained.) For the next 70 years, Hardy's book defined the first analysis course in Britain. Analysis texts could have borne titles like: 'Hardy made easier', 'An introduction to Hardy' or 'Hardy slimmed down'. Burkill's *First Course in Analysis* represents an outstanding example in the latter class.

In the last 40 years, Hardy's model has been put under strain from two different directions. The expansion of the university system has brought more students into mathematics, but the new students are less well prepared and less willing to study mathematics for its own sake. It is clear that 'Hardy diluted' cannot be appropriate for such students. On the other hand, the frontiers of mathematics have continued to advance and an analysis course for future researchers must prepare them to meet such things as manifolds and infinite dimensional spaces. Dieudonné's *Foundations of Modern Analysis* and Kolmogorov and Fomin's *Introductory Real Analysis* represent two very different but equally inspiring approaches to the problem.

As new topics enter the syllabus, old ones have to be removed. Today's best students get 'Hardy stripped to the bone' followed by a course on metric and topological spaces. They are taught speedily and efficiently, but some things have been lost. The TGV carries you swiftly across France, but isolates you from the land and its people. We claim to give our students the experience of mathematics, but provide plenty of 'routine exercises' and relegate 'the more difficult proofs' to appendices. Perhaps later generations of mathematicians may judge our teaching as harshly as Hardy and his generation judged the teaching of their Cambridge predecessors.

Hardy's book begins with a presentation of the properties of the real number system. In the first edition this is done axiomatically, but, in the second and later editions, Hardy constructs the real numbers starting from the rationals. Bertrand Russell says that there are two methods of presenting mathematics, the postulational and the constructive, with the postulational method having all the advantages of theft over honest toil, but a modern course would either leave the construction until much later or omit it altogether. Hardy allows the reader to skip the construction, but the reader should do at least some of the exercises that conclude the chapter. The next two chapters present material that the modern undergraduate would be expected to have met before embarking on a course of analysis.

The course proper starts with Chapter IV and V which introduce the notion of a limit. The treatment is is more leisurely than would be found in a modern introduction, but the reader who hurries through it is throwing away the advantage of listening to a great analyst talking about the elements of his subject. In one or two places the notation is definitely old-fashioned (as foreshadowed in the footnote in §71 *divergent* now means simply *not convergent*). It is easy, however, to make the transfer to modern notation later. More importantly, the reader should note that the theorems in §101 to §107 lie much deeper than those that precede them. Wherever Hardy makes use of the classes L and R of §17 he is appealing to the basic properties of the real numbers and the reader should pay close attention to the argument.

Chapter VI introduces differentiation and integration. Here, the most subtle arguments are to be found in §122 leading to the mean value theorem in §126. Until §161, Hardy considers integration as the inverse operation to differentiation (though he gives the link to area informally in §148) but in this section he defines the definite integral and completes his presentation of the foundations of the calculus. Once the foundations have been laid, he goes on to develop the standard methods and theorems of the calculus and give a rigorous account of the trigonometric, exponential and logarithmic function, both in the real and complex case.

In his *Mathematical Thought from Ancient to Modern Times* Kline dismisses the 100-year struggle of Bolzano, Cauchy, Abel, Dirichlet, Weierstrass, Cantor, Peano and others to rigorise analysis with the words 'the theorems of analysis only had to be more carefully formulated . . . all that rigour did was to substantiate what mathematicians had always known to be the case'. In fact, the rigorisation process revealed that several things that mathematicians had always known to be the case were, in fact, false. It is not true that every maximisation problem has a solution, it is not true that any continuous function must be differentiable except at a few exceptional points, it is not true that the boundary of a region is a negligible part of it, it is not true that every sufficiently smooth function is equal to its Taylor series, ...

Still more importantly, the process of rigourisation revealed the underlying structure of the real line and produced new tools (such as the Heine–Borel theorem of §106) to exploit that structure. In the decade that Hardy wrote his text, the study of the notion of area by Cantor, Peano, Jordan and Borel reached its apotheosis in the work of Lebesgue. Armed with the new tool of the Lebesgue integral and clear understanding of foundations, analysis entered on a golden century to which Hardy was to contribute such gems as the Hardy spaces and the Hardy–Littlewood maximal theorem.

> I wrote a great deal during . . . [1900–1910], but very little of any importance; there are not more than four or five papers which I can still remember with some satisfaction. The real crises of my career came in 1911, when I began my long collaboration with Littlewood, and in 1913 when I discovered Ramanujan.
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> [*A Mathematician's Apology*]

For its author and for his audience, *A Course in Pure Mathematics* represented not an end but a beginning.

Hardy published about 350 papers, including nearly 100 with Littlewood, but his contributions to mathematics did not stop there. He taught and inspired generations of research students. As one of them writes about Hardy's lectures, 'Whatever the subject was, he pursued it with an eager single-mindedness which the audience found irresistible. One felt, temporarily at least, that nothing else in the world but the proof of those theorems mattered. There could be no more inspiring director of the work of others. He was always at the head of a team of researchers, both colleagues and students, whom he provided with an inexhaustible stock of ideas on which to work.' Tichmarsh adds, and others confirm, that 'He was an extremely kind-hearted man, who could not bear any of his students to fail in their researches.'

Pólya recalled how Hardy '. . . valued clarity, yet what he valued most in mathematics was not clarity but power, surmounting great obstacles that others abandoned in despair.' Pólya also recalled how much Hardy loved jokes and told an anecdote which illustrated both aspects of Hardy's character.

> In working with Hardy, I once had an idea of which he approved. But afterwards I did not work sufficiently hard to carry out that idea, and Hardy disapproved. He did not tell me so, of course, yet it came out when he visited a zoological garden in Sweden with Marcel Riesz. In a cage there was a bear. The cage had a gate, and on the gate there was a lock. The bear sniffed at the lock, hit it with his paw, then growled a little, turned around and walked away. 'He is like Pólya', said Hardy. 'He has excellent ideas, but does not carry them out.'

In Hardy's presidential address to the London Mathematical Society in 1928 he was able to boast that he had sat through every word of every lecture of every meeting of every paper since he became secretary in 1917. He oversaw the foundation of the *Journal of the London Mathematical Society* and revived the *Quarterly Journal* at Oxford. The satisfactory state of the London Mathematical Society's finances today is the result of Hardy's bequest of a substantial fortune and the royalties of his books.

Hardy wrote or co-wrote several other classics. Perhaps the most remarkable is *Inequalities* with Littlewood and Pólya. In this book authors magically provide a coherent view of a subject which, though it lies at the heart of analysis, seems impossible to organise.

Hardy's *A Mathematician's Apology* is both a mathematical and a literary triumph and remains unequalled as a meditation on the life of a pure mathematician. It is also a defence of rationality and the free life of the intellect at a time when they were terribly threatened.

However, in my view, the most enchanting of his books is *Number Theory* (written with E. M. Wright). If I had to choose one book to take to a desert island, I would take Zygmund's *Trigonometric Series* if

I thought I might be rescued, but Hardy and Wright's *Number Theory* if I knew that I was never coming back.

To read Hardy is to read a mathematician fully aware of his own abilities but who treats you as a natural equal. May this book give as much pleasure to you as it has given to me.

T. W. Körner

PREFACE TO THE TENTH EDITION

THE changes in the present edition are as follows:

1. An index has been added. Hardy had begun a revision of an index compiled by Professor S. Mitchell; this has been completed, as far as possible on Hardy's lines, by Dr T. M. Flett.

2. The original proof of the Heine-Borel Theorem (pp. 197–199) has been replaced by two alternative proofs due to Professor A. S. Besicovitch.

3. Example 24, p. 394 has been added to.

August, 1950 J. E. LITTLEWOOD

PREFACE TO THE SEVENTH EDITION

THE changes in this edition are more important than in any since the second. The book has been reset, and this has given me the opportunity of altering it freely.

I have cancelled what was Appendix II (on the 'O, o, $\sim$' notation), and incorporated its contents in the appropriate places in the text. I have rewritten the parts of Chs. VI and VII which deal with the elementary properties of differential coefficients. Here I have found de la Vallée-Poussin's *Cours d'analyse* the best guide, and I am sure that this part of the book is much improved. These important changes have naturally involved many minor emendations.

I have inserted a large number of new examples from the papers for the Mathematical Tripos during the last twenty years, which should be useful to Cambridge students. These were collected for me by Mr E. R. Love, who has also read all the proofs and corrected many errors.

The general plan of the book is unchanged. I have often felt tempted, re-reading it in detail for the first time for twenty years, to make much more drastic changes both in substance and in style. It was written when analysis was neglected in Cambridge, and with an emphasis and enthusiasm which seem rather ridiculous now. If I were to rewrite it now I should not write (to use Prof. Littlewood's simile) like 'a missionary talking to cannibals', but with decent terseness and restraint; and, writing more shortly, I should be able to include a great deal more. The book would then be much more like a *Traité d'analyse* of the standard pattern.

It is perhaps fortunate that I have no time for such an undertaking, since I should probably end by writing a much better but much less individual book, and one less useful as an introduction to the books on analysis of which, even in England, there is now no lack.

November, 1937 G. H. H.

EXTRACT FROM THE PREFACE TO THE FIRST EDITION

THIS book has been designed primarily for the use of first year students at the Universities whose abilities reach or approach something like what is usually described as 'scholarship standard'. I hope that it may be useful to other classes of readers, but it is this class whose wants I have considered first. It is in any case a book for mathematicians: I have nowhere made any attempt to meet the needs of students of engineering or indeed any class of students whose interests are not primarily mathematical.

I regard the book as being really elementary. There are plenty of hard examples (mainly at the ends of the chapters): to these I have added, wherever space permitted, an outline of the solution. But I have done my best to avoid the inclusion of anything that involves really difficult ideas.

September, 1908 G. H. H.

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