LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor M. Reid, Mathematics Institute,
University of Warwick, Coventry, CV4 7AL, United Kingdom

The titles below are available from booksellers, or from Cambridge University Press at www.cambridge.org/mathematics

194 Independent random variables and rearrangement invariant spaces, M. BRAVERMAN
195 Arithmetic of blowup algebras, W. VASCONCELOS
196 Microlocal analysis for differential operators, A. GRIGIS & J. SJÖSTRAND
197 Two-dimensional homotopy and combinatorial group theory, C. HOG-ANGELONI *et al.*
198 The algebraic characterization of geometric 4-manifolds, J. A. HILLMAN
199 Invariant potential theory in the unit ball of $\boldsymbol{C}^n$, M. STOLL
200 The Grothendieck theory of dessins d'enfants, L. SCHNEPS (ed.)
201 Singularities, J.-P. BRASSELET (ed.)
202 The technique of pseudodifferential operators, H. O. CORDES
203 Hochschild cohomology of von Neumann algebras, A. SINCLAIR & R. SMITH
204 Combinatorial and geometric group theory, A. J. DUNCAN, N. D. GILBERT & J. HOWIE (eds)
205 Ergodic theory and its connections with harmonic analysis, K. PETERSEN & I. SALAMA (eds)
207 Groups of Lie type and their geometries, W. M. KANTOR & L. DI MARTINO (eds)
208 Vector bundles in algebraic geometry, N. J. HITCHIN, P. NEWSTEAD & W. M. OXBURY (eds)
209 Arithmetic of diagonal hypersurfaces over infite fields, F. Q. GOUVÉA & N. YUI
210 Hilbert C*-modules, E. C. LANCE
211 Groups 93 Galway / St Andrews I, C. M. CAMPBELL *et al.* (eds)
212 Groups 93 Galway / St Andrews II, C. M. CAMPBELL *et al.* (eds)
214 Generalised Euler–Jacobi inversion formula and asymptotics beyond all orders, V. KOWALENKO *et al.*
215 Number theory 1992–93, S. DAVID (ed.)
216 Stochastic partial differential equations, A. ETHERIDGE (ed.)
217 Quadratic forms with applications to algebraic geometry and topology, A. PFISTER
218 Surveys in combinatorics, 1995, P. ROWLINSON (ed.)
220 Algebraic set theory, A. JOYAL & I. MOERDIJK
221 Harmonic approximation, S. J. GARDINER
222 Advances in linear logic, J.-Y. GIRARD, Y. LAFONT & L. REGNIER (eds)
223 Analytic semigroups and semilinear initial boundary value problems, KAZUAKI TAIRA
224 Computability, enumerability, unsolvability, S. B. COOPER, T. A. SLAMAN & S. S. WAINER (eds)
225 A mathematical introduction to string theory, S. ALBEVERIO *et al.*
226 Novikov conjectures, index theorems and rigidity I, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
227 Novikov conjectures, index theorems and rigidity II, S. FERRY, A. RANICKI & J. ROSENBERG (eds)
228 Ergodic theory of $\boldsymbol{Z}^d$ actions, M. POLLICOTT & K. SCHMIDT (eds)
229 Ergodicity for infinite dimensional systems, G. DA PRATO & J. ZABCZYK
230 Prolegomena to a middlebrow arithmetic of curves of genus 2, J. W. S. CASSELS & E. V. FLYNN
231 Semigroup theory and its applications, K. H. HOFMANN & M. W. MISLOVE (eds)
232 The descriptive set theory of Polish group actions, H. BECKER & A. S. KECHRIS
233 Finite fields and applications, S. COHEN & H. NIEDERREITER (eds)
234 Introduction to subfactors, V. JONES & V. S. SUNDER
235 Number theory 1993–94, S. DAVID (ed.)
236 The James forest, H. FETTER & B. G. DE BUEN
237 Sieve methods, exponential sums, and their applications in number theory, G. R. H. GREAVES *et al.*
238 Representation theory and algebraic geometry, A. MARTSINKOVSKY & G. TODOROV (eds)
240 Stable groups, F. O. WAGNER
241 Surveys in combinatorics, 1997, R. A. BAILEY (ed.)
242 Geometric Galois actions I, L. SCHNEPS & P. LOCHAK (eds)
243 Geometric Galois actions II, L. SCHNEPS & P. LOCHAK (eds)
244 Model theory of groups and automorphism groups, D. EVANS (ed.)
245 Geometry, combinatorial designs and related structures, J. W. P. HIRSCHFELD *et al.*
246 p-Automorphisms of finite p-groups, E. I. KHUKHRO
247 Analytic number theory, Y. MOTOHASHI (ed.)
248 Tame topology and o-minimal structures, L. VAN DEN DRIES
249 The atlas of finite groups: ten years on, R. CURTIS & R. WILSON (eds)
250 Characters and blocks of finite groups, G. NAVARRO
251 Gröbner bases and applications, B. BUCHBERGER & F. WINKLER (eds)
252 Geometry and cohomology in group theory, P. KROPHOLLER, G. NIBLO, R. STÖHR (eds)
253 The q-Schur algebra, S. DONKIN
254 Galois representations in arithmetic algebraic geometry, A. J. SCHOLL & R. L. TAYLOR (eds)
255 Symmetries and integrability of difference equations, P. A. CLARKSON & F. W. NIJHOFF (eds)
256 Aspects of Galois theory, H. VÖLKLEIN *et al.*
257 An introduction to noncommutative differential geometry and its physical applications 2ed, J. MADORE
258 Sets and proofs, S. B. COOPER & J. TRUSS (eds)
259 Models and computability, S. B. COOPER & J. TRUSS (eds)
260 Groups St Andrews 1997 in Bath, I, C. M. CAMPBELL *et al.*
261 Groups St Andrews 1997 in Bath, II, C. M. CAMPBELL *et al.*
262 Analysis and logic, C. W. HENSON, J. IOVINO, A. S. KECHRIS & E. ODELL
263 Singularity theory, B. BRUCE & D. MOND (eds)
264 New trends in algebraic geometry, K. HULEK, F. CATANESE, C. PETERS & M. REID (eds)
265 Elliptic curves in cryptography, I. BLAKE, G. SEROUSSI & N. SMART
267 Surveys in combinatorics, 1999, J. D. LAMB & D. A. PREECE (eds)
268 Spectral asymptotics in the semi-classical limit, M. DIMASSI & J. SJÖSTRAND
269 Ergodic theory and topological dynamics, M. B. BEKKA & M. MAYER
270 Analysis on Lie Groups, N. T. VAROPOULOS & S. MUSTAPHA
271 Singular perturbations of differential operators, S. ALBEVERIO & P. KURASOV
272 Character theory for the odd order theorem, T. PETERFALVI
273 Spectral theory and geometry, E. B. DAVIES & Y. SAFAROV (eds)
274 The Mandelbrot set, theme and variations, TAN LEI (ed.)
275 Descriptive set theory and dynamical systems, M. FOREMAN *et al.*

276 Singularities of plane curves, E. CASAS-ALVERO
277 Computational and geometric aspects of modern algebra, M. D. ATKINSON *et al*.
278 Global attractors in abstract parabolic problems, J. W. CHOLEWA & T. DLOTKO
279 Topics in symbolic dynamics and applications, F. BLANCHARD, A. MAASS & A. NOGUEIRA (eds)
280 Characters and automorphism groups of compact Riemann surfaces, T. BREUER
281 Explicit birational geometry of 3-folds, A. CORTI & M. REID (eds)
282 Auslander–Buchweitz approximations of equivariant modules, M. HASHIMOTO
283 Nonlinear elasticity, Y. FU & R. OGDEN (eds)
284 Foundations of computational mathematics, R. DEVORE, A. ISERLES & E. SÜLI (eds)
285 Rational points on curves over finite fields, H. NIEDERREITER & C. XING
286 Clifford algebras and spinors 2ed, P. LOUNESTO
287 Topics on Riemann surfaces and Fuchsian groups, E. BUJALANCE *et al.*
288 Surveys in combinatorics, 2001, J. HIRSCHFELD (ed.)
289 Aspects of Sobolev-type inequalities, L. SALOFF-COSTE
290 Quantum groups and Lie theory, A. PRESSLEY (ed.)
291 Tits buildings and the model theory of groups, K. TENT (ed.)
292 A quantum groups primer, S. MAJID
293 Second order partial differential equations in Hilbert spaces, G. DA PRATO & J. ZABCZYK
294 Introduction to operator space theory, G. PISIER
295 Geometry and Integrability, L. MASON & YAVUZ NUTKU (eds)
296 Lectures on invariant theory, I. DOLGACHEV
297 The homotopy category of simply connected 4-manifolds, H.-J. BAUES
298 Higher operands, higher categories, T. LEINSTER
299 Kleinian Groups and Hyperbolic 3-Manifolds, Y. KOMORI, V. MARKOVIC & C. SERIES (eds)
300 Introduction to Möbius Differential Geometry, U. HERTRICH-JEROMIN
301 Stable Modules and the D(2)-Problem, F. E. A. JOHNSON
302 Discrete and Continuous Nonlinear Schrödinger Systems, M. J. ABLOWITZ, B. PRINARI & A. D. TRUBATCH
303 Number Theory and Algebraic Geometry, M. REID & A. SKOROBOGATOV (eds)
304 Groups St Andrews 2001 in Oxford Vol. 1, C. M. CAMPBELL, E. F. ROBERTSON & G. C. SMITH (eds)
305 Groups St Andrews 2001 in Oxford Vol. 2, C. M. CAMPBELL, E. F. ROBERTSON & G. C. SMITH (eds)
306 Peyresq lectures on geometric mechanics and symmetry, J. MONTALDI & T. RATIU (eds)
307 Surveys in Combinatorics 2003, C. D. WENSLEY (ed.)
308 Topology, geometry and quantum field theory, U. L. TILLMANN (ed.)
309 Corings and Comdules, T. BRZEZINSKI & R. WISBAUER
310 Topics in Dynamics and Ergodic Theory, S. BEZUGLYI & S. KOLYADA (eds)
311 Groups: topological, combinatorial and arithmetic aspects, T. W. MÜLLER (ed.)
312 Foundations of Computational Mathematics, Minneapolis 2002, FELIPE CUCKER *et al*. (eds)
313 Transcendantal aspects of algebraic cycles, S. MÜLLER-STACH & C. PETERS (eds)
314 Spectral generalizations of line graphs, D. CVETKOVIC, P. ROWLINSON & S. SIMIC
315 Structured ring spectra, A. BAKER & B. RICHTER (eds)
316 Linear Logic in Computer Science, T. EHRHARD *et al*. (eds)
317 Advances in elliptic curve cryptography, I. F. BLAKE, G. SEROUSSI & N. SMART
318 Perturbation of the boundary in boundary-value problems of Partial Differential Equations, DAN HENRY
319 Double Affine Hecke Algebras, I. CHEREDNIK
320 L-Functions and Galois Representations, D. BURNS, K. BUZZARD & J. NEKOVÁŘ (eds)
321 Surveys in Modern Mathematics, V. PRASOLOV & Y. ILYASHENKO (eds)
322 Recent perspectives in random matrix theory and number theory, F. MEZZADRI, N. C. SNAITH (eds)
323 Poisson geometry, deformation quantisation and group representations, S. GUTT *et al*. (eds)
324 Singularities and Computer Algebra, C. LOSSEN & G. PFISTER (eds)
325 Lectures on the Ricci Flow, P. TOPPING
326 Modular Representations of Finite Groups of Lie Type, J. E. HUMPHREYS
328 Fundamentals of Hyperbolic Manifolds, R. D. CANARY, A. MARDEN & D. B. A. EPSTEIN (eds)
329 Spaces of Kleinian Groups, Y. MINSKY, M. SAKUMA & C. SERIES (eds)
330 Noncommutative Localization in Algebra and Topology, A. RANICKI (ed.)
331 Foundations of Computational Mathematics, Santander 2005, L. PARDO, A. PINKUS, E. SULI & M. TODD (eds)
332 Handbook of Tilting Theory, L. ANGELERI HÜGEL, D. HAPPEL & H. KRAUSE (eds)
333 Synthetic Differential Geometry 2ed, A. KOCK
334 The Navier–Stokes Equations, P. G. DRAZIN & N. RILEY
335 Lectures on the Combinatorics of Free Probability, A. NICA & R. SPEICHER
336 Integral Closure of Ideals, Rings, and Modules, I. SWANSON & C. HUNEKE
337 Methods in Banach Space Theory, J. M. F. CASTILLO & W. B. JOHNSON (eds)
338 Surveys in Geometry and Number Theory, N. YOUNG (ed.)
339 Groups St Andrews 2005 Vol. 1, C. M. CAMPBELL, M. R. QUICK, E. F. ROBERTSON & G. C. SMITH (eds)
340 Groups St Andrews 2005 Vol. 2, C. M. CAMPBELL, M. R. QUICK, E. F. ROBERTSON & G. C. SMITH (eds)
341 Ranks of Elliptic Curves and Random Matrix Theory, J. B. CONREY, D. W. FARMER, F. MEZZADRI & N. C. SNAITH (eds)
342 Elliptic Cohomology, H. R. MILLER & D. C. RAVENEL (eds)
343 Algebraic Cycles and Motives Vol. 1, J. NAGEL & C. PETERS (eds)
344 Algebraic Cycles and Motives Vol. 2, J. NAGEL & C. PETERS (eds)
345 Algebraic and Analytic Geometry, A. NEEMAN
346 Surveys in Combinatorics, 2007, A. HILTON & J. TALBOT (eds)
347 Surveys in Contemporary Mathematics, N. YOUNG & Y. CHOI (eds)
348 Transcendental Dynamics and Complex Analysis, P. J. RIPPON & G. M. STALLARD (eds)
349 Model Theory with Applications to Algebra and Analysis Vol 1, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds)
350 Model Theory with Applications to Algebra and Analysis Vol 2, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds)
351 Finite von Neumann Algebras and Masas, A. SINCLAIR & R. SMITH
352 Number Theory and Polynomials, J. MCKEE & C. SMYTH (eds)
353 Trends in Stochastic Analysis, J. BLATH, P. MÖRTERS & M. SCHEUTZOW (eds)
354 Groups and Analysis, K. TENT (ed)
355 Non-equilibrium Statistical Mechanics and Turbulence, J. CARDY, G. FALKOVICH & K. GAWEDZKI, with S. NAZARENKO & O. V. ZABORONSKI (eds)
356 Elliptic Curves and Big Galois Representations, D. DELBOURGO

London Mathematical Society Lecture Note Series: 357

Algebraic Theory of Differential Equations

Edited by

MALCOLM A. H. MACCALLUM
Queen Mary, University of London

ALEXANDER V. MIKHAILOV
University of Leeds

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9780521720083

First published 2009

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-72008-3 Paperback

Contents

Preface

This book presents lectures given during a school and workshop organized at Heriot-Watt University in July and August 2006, with funding from the London Mathematical Society (LMS), which supported the Invited Lectures of Michael Singer, and the International Centre for Mathematical Sciences (ICMS) in Edinburgh, which supported the Workshop.

The origin of these events lies in a suggestion made by one of us (AVM) to the other that we should make a proposal for an Isaac Newton Institute programme. After some thought we concluded that a smaller scale proposal would be more appropriate to the present state of the field and the wishes of likely participants, as well as our capacity to write proposals.

We therefore, with his agreement, proposed Michael Singer as the speaker for the 2006 LMS Invited Lecture series, and simultaneously make a proposal to ICMS for a workshop to follow the Invited Lectures: happily both proposals were accepted. We were very glad that in making the workshop proposal we had the assistance of Michael himself and of Sergei Tsarev, who were very helpful in suggesting topics for inclusion and people to invite, whether or not they became speakers.

Between the initial proposal and carrying it out we were fortunate to recruit Chris Eilbeck of Heriot-Watt as an additional organizer. His local knowledge and connections with ICMS were invaluable in actually mounting the event, and it is hard to express our thanks forcefully enough!

Michael Singer is an excellent lecturer of great clarity: from personal experience we also knew that he is generous with his time and ideas, especially to new workers in the field. He has made outstanding contributions in Differential Algebra and Differential (and Difference) Galois Theory, and naturally it was that which we asked him to lecture on. We

also had in mind from the start that we should complement Michael's lectures with some shorter series of lectures on other topics in the general field, which, together with Michael's longer series, would enable people new to the field to profit from the workshop. In a way, we were lucky that because he had recently published books in the field, Michael did not wish to write up the lectures at book length, as many LMS Invited Lecturers have, and we are therefore able to present the work of other lecturers as well.

We aimed, in both the school and the workshop, to cover the full range of algebraic approaches to differential equations: differential algebra, D-modules, model-theoretical aspects of the theory of differential equations, the Inverse Transform method for nonlinear partial differential equations, and the algebraic theory of factorization of differential equations and systems of differential equations. The only major approach which may have been under-represented was Lie symmetry methods and their generalizations, on which there are already excellent texts. We were also particularly keen to encourage and facilitate interactions between the various approaches, which had not really been brought together before.

As it turned out we were able to mount a good many of the subsidiary shorter lecture series we hoped for, with speakers distinguished for both content and presentation. As in the school itself, the major contribution in this book is Singer's and is an expanded written version of his lectures, but we also have written versions of most of the others. Anand Pillay (and others working in model theory and related areas) had written a number of other expository and research papers about the same time, and so did not wish to make a full writeup, but Anand nevertheless kindly agreed to write a short account referring to those other articles, while the general area covered by Vladimir Sokolov's lectures is discussed in the contribution of Mikhailov, Wang and Novikov.

In general we felt that the workshop talks, although covering a lot of very interesting and novel work, were too specialized for a lecture notes volume, but we invited some of those whose talks had more of a overview character, or had included a section of that kind, to contribute. Felix Ulmer and Jacques-Arthur Weil provided a combined contribution, Wang and Novikov joined theirs with Mikhailov's, and Hietarinta also kindly agreed to contribute.

We heartily thank our co-organizers, without whom the event would have been poorer both intellectually and practically, our sponsors, LMS and ICMS, the lecturers whose work is presented here, and the many

other contributors to the workshop. We also gratefully acknowledge additional support from the Edinburgh Mathematical Society and the Royal Society of Edinburgh.

Finally, we would like to thank the ICMS very much for its organizational help, as well as the funding, in particular its Director, John Toland, and the administrative staff, Tracey Dart, Morag Burton and Audrey Brown, who made all the practical arrangements including coping with the knock-on effects of the sudden change in airline baggage regulations that happened at the end of the second week, and to thank the Conference staff of Heriot-Watt for enabling such a pleasant and stimulating two weeks. In recognition of the work of ICMS, any royalties from this book will be given to them to add to student support at future events.

Malcolm MacCallum
Alexander Mikhailov
January 2008