Analysis on Lie Groups
An Introduction

The subject of analysis on Lie groups comprises an eclectic group of topics which can be treated from many different perspectives. This self-contained text concentrates on the perspective of analysis to the topics and methods of non-commutative harmonic analysis, assuming only elementary knowledge of linear algebra and basic differential calculus.

The author avoids non-essential technical discussion and instead describes in detail many interesting examples, including formulae which have not previously appeared in book form. Topics covered include the Haar measure and invariant integration, spherical harmonics, Fourier analysis and the heat equation, the Poisson kernel, the Laplace equation and harmonic functions.

Perfect for advanced undergraduates and graduates in geometric analysis, harmonic analysis and representation theory, the tools developed will also be useful for specialists in stochastic calculation and statistics. With numerous exercises and worked examples, the text is ideal for a graduate course on analysis on Lie groups.

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An Introduction

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Preface

This book stems from notes of a master's course given at the *Université Pierre et Marie Curie*. This is an introduction to the theory of Lie groups and to the study of their representations, with applications to analysis. In this introductory text we do not present the general theory of Lie groups, which assumes a knowledge of differential manifolds. We restrict ourself to linear Lie groups, that is groups of matrices. The tools used to study these groups come mainly from linear algebra and differential calculus. A linear Lie group is defined as a closed subgroup of the linear group $GL(n, \mathbb{R})$. The exponential map makes it possible to associate to a linear Lie group its Lie algebra, which is a subalgebra of the algebra of square matrices $M(n, \mathbb{R})$ endowed with the bracket $[X, Y] = XY - YX$. Then one can show that every linear Lie group is a manifold embedded in the finite dimensional vector space $M(n, \mathbb{R})$. This is an advantage of the definition we give of a linear Lie group, but it is worth noticing that, according to this definition, not every Lie subalgebra of $M(n, \mathbb{R})$ is the Lie algebra of a linear Lie group, that is a closed subgroup of $GL(n, \mathbb{R})$. The Haar measure of a linear Lie group is built in terms of differential forms, and these are used to establish several integration formulae, linking geometry and analysis. The basic properties of irreducible representations of compact groups, that is the Peter–Weyl Theory, are first presented in a general setting, then described explicitly in the case of the simplest non-commutative compact Lie groups: the special unitary group $SU(2)$, and the special orthogonal group $SO(3)$, then further in the case of the unitary groups $U(n)$. The topics in analysis we present are centred on a basic object: the Laplace operator. Fourier analysis on a compact linear Lie group provides a diagonalisation of the Laplace operator, and the Fourier method is in particular a natural method for solving the Cauchy problem for the heat equation on the group $SU(2)$. Similarly, analysis on the sphere in $\mathbb{R}^n$ uses the spherical harmonic decomposition and makes clear the interaction which exists between the orthogonal group $O(n)$ and Fourier analysis on $\mathbb{R}^n$, shown for instance by the Bochner–Hecke relations and potential theory, while expanding a harmonic function as a series of harmonic homogeneous polynomials. Questions of the same nature arise as one considers the action of the orthogonal group $O(n)$ on the space $Sym(n, \mathbb{R})$ of real symmetric matrices, or the action of the unitary group $U(n)$ on the space $Herm(n, \mathbb{C})$ of Hermitian matrices. The formula for the radial part of the Laplace operator plays an important role; in particular, it leads to evaluation of integral orbitals via solution of the Cauchy problem for the heat equation on the space $Herm(n, \mathbb{C})$ of Hermitian matrices. To study the irreducible representations of the unitary group $U(n)$ we start from the highest weight theorem. This is a special case of the Weyl theory of irreducible representations of compact Lie groups. The characters of the irreducible representations of the unitary group are expressed in terms of Schur functions, for which we establish some combinatoric properties. These make it possible to write explicit Fourier expansions of some central functions, and also general Taylor expansions for holomorphic functions of a matrix argument.

The invariant analysis topics we are dealing with in this book illustrate how Lie groups are involved in many fields: matrix analysis, Fourier analysis, complex analysis, mathematical physics.

Each chapter is followed by numerous exercises. Some topics which are not treated in the text are introduced as problems. For example, in Chapter 7, we present the construction of an equivariant isomorphism between the space of polynomials in two variables, homogeneous of degree 2ℓ, and the space of harmonic polynomials in three variables, homogeneous of degree ℓ.

Many books deal with the theory of Lie groups. We cite several of them in the bibliography. We were inspired at several points by the presentation given by J. Hilgert and K.-H. Neeb in their book *Lie-Gruppen und Lie-Algebren*, and we have included elegant arguments from the book by R. Mneimné and F. Testard, *Introduction à la Théorie des Groupes de Lie Classiques*.

We thank Rached Mneimné, Hervé Sabourin, and Valerio Toledano for reading and commenting on preliminary versions of this text, and giving us valuable advice for improving it.