1
The linear group

The linear group $GL(n, \mathbb{R})$ is the group of invertible real $n \times n$ matrices. After some topological preliminaries we present some subgroups of the linear group which play an important role in geometry and analysis. We establish the polar and Gram decompositions, which will be useful for proving some topological properties of these groups.

1.1 Topological groups

A topological group is a group equipped with a topology such that the maps

$$(x, y) \mapsto xy, \quad G \times G \to G,$$
$$x \mapsto x^{-1}, \quad G \to G,$$

are continuous. This amounts to saying that the map

$$(x, y) \mapsto xy^{-1}, \quad G \times G \to G$$

is continuous.

A topological group is Haussdorff if $\{e\}$ is closed (e is the identity element of G).

Let H be a subgroup of a topological group G. If H is open then H is also closed. In fact, if $g \notin H$, gH is a neighbourhood of g contained in H^c, therefore H^c is open.

Let G_0 be the connected component of e in G (one says the *identity component*). Then G_0 is a normal subgroup of G. In fact, if $g \in G_0$, then $g^{-1}G_0$ is connected and contains e, hence $g^{-1}G_0 \subset G_0$, and G_0 is a subgroup of G. Furthermore, if $g \in G$, then gG_0g^{-1} is connected and contains e, hence $gG_0g^{-1} \subset G_0$, and G_0 is a normal subgroup.

Proposition 1.1.1 *Let V be a connected neighbourhood of e in a topological group G, then*

$$\bigcup_{n=1}^{\infty} V^n = G_0,$$

where G_0 denotes the identity component of G.

Hence a connected topological group is generated by any neighbourhood of the identity element.

Proof. In fact, if V is a neighbourhood of e, then the increasing union $U = \cup_{n=1}^{\infty} V^n$ is an open set since V^{n+1} is a neighbourhood of each point of V^n.

If V is connected then U is connected as well since it is a union of connected sets, all of which contain e. Therefore $U \subset G_0$. Let $W = V \cap V^{-1}$, then

$$U' = \bigcup_{n=1}^{\infty} W^n$$

is an open subgroup of G, hence closed. Since $U' \subset G_0$, because $U' \subset U$, then $U' = G_0$, and therefore $U = G_0$. □

The topology of a topological group is determined by the set $\mathcal{V}$ of the neighbourhoods of e. This set has the following properties.

(a) If $V \in \mathcal{V}$, there exists V_1 and $V_2 \in \mathcal{V}$ such that $V_1 \cdot V_2 \subset V$.
(b) If $V \in \mathcal{V}$, then $V^{-1} \in \mathcal{V}$.
(c) Let $V \in \mathcal{V}$ and $g \in G$, then $gVg^{-1} \in \mathcal{V}$.

Conversely, if G is a group and if $\mathcal{V}$ is a family of non-empty subsets of G with the following properties:

every subset of G which contains a subset of $\mathcal{V}$ belongs to $\mathcal{V}$,
any finite intersection of subsets of $\mathcal{V}$ belongs to $\mathcal{V}$,

(i.e. $\mathcal{V}$ is a filter), and also properties (a), (b), and (c), then there exists a unique topology for which G is a topological group such that $\mathcal{V}$ is the family of the neighbourhoods of e.

The neighbourhoods of an element $g \in G$ are the subsets gV $(V \in \mathcal{V})$.

1.2 The group $GL(n, \mathbb{R})$

Let $M(n, \mathbb{R})$ denote the algebra of $n \times n$ matrices with entries in $\mathbb{R}$, and $GL(n, \mathbb{R})$ the group of invertible matrices in $M(n, \mathbb{R})$, which is called the *linear group*. We will consider this group from the viewpoints of topology and differential calculus.

We consider on $\mathbb{R}^n$ the Euclidean norm

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

associated to the Euclidean inner product

$$(x|y) = x_1 y_1 + \cdots + x_n y_n,$$

and on $M(n, \mathbb{R})$ the norm

$$\|A\| = \sup_{x \in \mathbb{R}^n, \|x\| \leq 1} \|Ax\|.$$

Let us recall that, on a finite dimensional vector space, all the norms are equivalent. Note that the norm we consider on $M(n, \mathbb{R})$ is an algebra norm:

$$\|AB\| \leq \|A\|\|B\|.$$

One can check that the product on $M(n, \mathbb{R})$ is a continuous map.

Proposition 1.2.1 *The group $GL(n, \mathbb{R})$ is open in $M(n, \mathbb{R})$. The map $g \mapsto g^{-1}$, from $GL(n, \mathbb{R})$ onto itself, is continuous.*

Proof. One can prove this proposition using Cramer's formulae. In fact,

$$GL(n, \mathbb{R}) = \{g \in M(n, \mathbb{R}) \mid \det(g) \neq 0\},$$

and

$$g^{-1} = \frac{1}{\det g} \tilde{g},$$

where $\tilde{g}$ denotes the cofactor matrix whose entries are polynomials in the entries of g. We will give another proof which holds if, instead of $M(n, \mathbb{R})$, one considers any Banach algebra, possibly infinite dimensional.

(a) *Let $M \in M(n, \mathbb{R})$. If $\|M\| < 1$, then $I + M$ is invertible and*

$$\|(I + M)^{-1}\| \leq \frac{1}{1 - \|M\|}.$$

In fact, the series $\sum_{k=0}^{\infty} (-1)^k M^k$ converges in norm and its sum is equal to $(I + M)^{-1}$:

$$(I + M)^{-1} = \sum_{k=0}^{\infty} (-1)^k M^k.$$

Furthermore,

$$\|(I+M)^{-1}\| \leq \sum_{k=0}^{\infty} \|M^k\| \leq \sum_{k=0}^{\infty} \|M\|^k = \frac{1}{1-\|M\|}.$$

(b) *Let A be an invertible matrix. If B is a matrix such that*

$$\|B-A\| < \|A^{-1}\|^{-1},$$

then B is invertible, and if $\|B-A\| \leq \varepsilon < \|A^{-1}\|^{-1}$,

$$\|B^{-1}-A^{-1}\| \leq \frac{\|A^{-1}\|^2\varepsilon}{1-\|A^{-1}\|\varepsilon}.$$

One can write

$$B = A\big(I + A^{-1}(B-A)\big),$$

and one applies (a) to $M = A^{-1}(B-A)$. Note that $\|M\| \leq \|A^{-1}\|\varepsilon$. Therefore, if $\varepsilon < \|A^{-1}\|^{-1}$, then $I+M$ is invertible and

$$B^{-1} = (I+M)^{-1}A^{-1}, \quad \|B^{-1}\| \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\varepsilon}.$$

Furthermore,

$$B^{-1}-A^{-1} = B^{-1}(A-B)A^{-1},$$

hence

$$\|B^{-1}-A^{-1}\| \leq \frac{\|A^{-1}\|^2\varepsilon}{1-\varepsilon\|A^{-1}\|}. \qquad \square$$

Therefore we can state the following.

Theorem 1.2.2 *The group $GL(n,\mathbb{R})$, equipped with the topology inherited from $M(n,\mathbb{R})$, is a topological group.*

From now on $GL(n,\mathbb{R})$ will denote this topological group.

Proposition 1.2.3 *The subsets*

$$\{g \in GL(n,\mathbb{R}) \mid \|g\| \leq C,\ \|g^{-1}\| \leq C\},$$

where C is a constant, are compact, and every compact subset of $GL(n,\mathbb{R})$ is contained in a subset of that form.

Proof. Let us show that the subset

$$Q = \{g \in GL(n,\mathbb{R}) \mid \|g\| \leq C,\ \|g^{-1}\| \leq C\}$$

is compact. Let (g_n) be a sequence of elements in Q. Since a closed ball in $M(n,\mathbb{R})$ is compact, it is possible to extract from the sequence (g_n) a subsequence (g_{n_k}) which converges to an element g in $M(n,\mathbb{R})$ with $\|g\| \leq C$. Since $\|g_{n_k}^{-1}\| \leq 1$, it is possible to extract from the sequence $(g_{n_k}^{-1})$ a subsequence which converges to an element h in $M(n,\mathbb{R})$ with $\|h\| \leq C$. Furthermore, for every n, $g_n g_n^{-1} = I$, hence $gh = I$ or $h = g^{-1}$, $g \in GL(n,\mathbb{R})$, and $g \in E$. □

Note that the group $GL(n,\mathbb{R})$ is equal to the increasing sequence of the compact subsets

$$Q_k = \left\{ g \in GL(n,\mathbb{R}) \mid \|g\| \leq k,\ |\det g| \geq \frac{1}{k} \right\} \quad (k \in \mathbb{N}^*).$$

1.3 Examples of subgroups of $GL(n,\mathbb{R})$

(a) Let $SL(n,\mathbb{R})$ denote the *special linear group* defined by

$$SL(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) \mid \det g = 1\}.$$

It is a closed subgroup of $GL(n,\mathbb{R})$ which is normal because it is the kernel of the continuous group morphism

$$\det : \ GL(n,\mathbb{R}) \to \mathbb{R}^*.$$

(b) Let $O(n)$ denote the *orthogonal group* defined by

$$O(n) = \{g \in GL(n,\mathbb{R}) \mid \forall x \in \mathbb{R}^n,\ \|gx\| = \|x\|\}.$$

By polarising one can show that $g \in O(n)$ if and only if

$$\forall x, y \in \mathbb{R}^n, \quad (gx|gy) = (x|y),$$

and this can be written

$$g^T g = I, \quad \text{or } g^{-1} = g^T,$$

where g^T denotes the transposed matrix of g. Therefore, if $g \in O(n)$, then $\det g = \pm 1$.

The rows of $g \in O(n)$ are orthogonal unit vectors, and the same holds for the columns. The subgroup $O(n)$ is a compact subgroup of $GL(n,\mathbb{R})$. This follows from Proposition 1.2.3. In fact, for g in $O(n)$,

$$\|g\| = 1, \quad \|g^{-1}\| = 1.$$

Cambridge University Press
978-0-521-71930-8 - Analysis on Lie Groups: An Introduction
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The *special orthogonal group* $SO(n)$ is the subgroup of orthogonal matrices with determinant equal to one:

$$SO(n) = O(n) \cap SL(n, \mathbb{R}).$$

(c) More generally, let us consider a non-degenerate bilinear form b on $\mathbb{R}^n$, and the subgroup $O(b)$ defined by

$$O(b) = \{g \in GL(n, \mathbb{R}) \mid \forall x, y \in \mathbb{R}^n,\ b(gx, gy) = b(x, y)\}.$$

Let B be the matrix of the bilinear form b:

$$b(x, y) = y^T Bx.$$

The condition $g \in O(b)$ can be written

$$g^T Bg = B.$$

Let us observe that, since the matrix B is invertible, this condition implies that g is invertible. The subgroup $O(b)$ is closed in $GL(n, \mathbb{R})$ and, for $g \in O(b)$,

$$g^{-1} = B^{-1} g^T B.$$

If b is the symmetric bilinear form

$$b(x, y) = \sum_{i=1}^{p} x_i y_i - \sum_{i=1}^{q} x_{p+i} y_{p+i}, \qquad p + q = n,$$

then one can write $O(b) = O(p, q)$:

$$O(p, q) = \{g \in GL(n, \mathbb{R}) \mid g^T I_{p,q} g = I_{pq}\},$$

where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

The subgroup $O(p, q)$ is called the *pseudo-orthogonal group*.

If b is a symmetric bilinear form with signature (p, q), there exists $g_0 \in GL(n, \mathbb{R})$ such that $B = g_0^T I_{p,q} g_0$. (This is Sylvester's law of inertia.) Therefore, the subgroup $O(b)$ is conjugate to $O(p, q)$:

$$O(b) = g_0^{-1} O(p, q) g_0.$$

The subgroup $O(1, 3)$ plays an important role in relativity theory. This is in fact the group of linear transformations of space-time $\mathbb{R}^4$ which preserve the Lorentz form

$$t^2 - x^2 - y^2 - z^2.$$

(d) Another important example is the case of a non-degenerate skewsymmetric bilinear form. Such a form only exists if n is even, $n = 2m$, and then there exists a basis with respect to which

$$b(x, y) = -\sum_{i=1}^{m} x_i y_{m+i} + \sum_{i=1}^{m} x_{m+i} y_i.$$

The matrix of this form is

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

In this case the subgroup $O(b)$ is the *symplectic group*

$$Sp(m, \mathbb{R}) = \{g \in GL(2m, \mathbb{R}) \mid g^T J g = J\}.$$

(e) Let us mention the group of upper triangular matrices:

$$\mathrm{T}(n, \mathbb{R}) = \{g \in GL(n, \mathbb{R}) \mid g_{ij} = 0 \text{ if } i > j\},$$

which is called the *upper triangular group*. We also have the *strict upper triangular group*:

$$\mathrm{T}_0(n, \mathbb{R}) = \{g \in GL(n, \mathbb{R}) \mid g_{ij} = 0 \text{ if } i > j, \text{ and } g_{ii} = 1\}.$$

One can check that $\mathrm{T}_0(n, \mathbb{R})$ is a normal subgroup of $\mathrm{T}(n, \mathbb{R})$.

(f) Consider on $\mathbb{C}^n$ the Hermitian inner product

$$(x|y) = \sum_{i=1}^{n} x_i \bar{y}_i.$$

The *unitary group* $U(n)$ is the subgroup of $GL(n, \mathbb{C})$ consisting of matrices which preserve this inner product. This can be written

$$U(n) = \{g \in GL(n, \mathbb{C}) \mid g^* g = I\}.$$

The *special unitary group* $SU(n)$ is the group of unitary matrices with determinant one. The *pseudo-unitary group* $U(p, q)$ is defined as

$$U(p, q) = \{g \in GL(n, \mathbb{C}) \mid g^* I_{p,q} g = I_{p,q}\}.$$

1.4 Polar decomposition in $GL(n, \mathbb{R})$

Let us denote by $\mathcal{P}_n$ the set of positive definite real symmetric $n \times n$ matrices. This is an open convex cone in the vector space $Sym(n, \mathbb{R})$ of real symmetric

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Cambridge University Press
978-0-521-71930-8 - Analysis on Lie Groups: An Introduction
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matrices. One can see that $\mathcal{P}_n$ is open as follows. To a matrix $p \in \mathcal{P}_n$ one associates the quadratic form

$$Q(x) = (px|x).$$

The function Q is continuous on the unit sphere S of $\mathbb{R}^n$. It is strictly positive and, since S is compact,

$$\alpha := \inf_{x \in S} Q(x) > 0.$$

One can show that the open ball with centre p and radius α is contained in $\mathcal{P}_n$.

Theorem 1.4.1 (Polar decomposition) *Every $g \in GL(n, \mathbb{R})$ decomposes uniquely as*

$$g = kp,$$

with $k \in O(n)$, $p \in \mathcal{P}_n$. Furthermore the map

$$O(n) \times \mathcal{P}_n \to GL(n, \mathbb{R}), \quad (k, p) \mapsto g = kp,$$

is a homeomorphism

Proof. (a) *Existence.* Let $g \in GL(n, \mathbb{R})$. If $x \neq 0$ then

$$(g^T gx|x) = \|gx\|^2 > 0,$$

therefore $A = g^T g \in \mathcal{P}_n$. It follows that the symmetric matrix A, which is diagonalisable in an orthogonal basis:

$$A = h \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} h^{-1} \quad (h \in O(n)),$$

has positive eigenvalues λ_i. The matrix

$$p = h \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} h^{-1}$$

belongs to $\mathcal{P}_n$, and $p^2 = A$. Define

$$k = gp^{-1},$$

then

$$k^T k = p^{-1} g^T g p^{-1} = p^{-1} A p^{-1} = I,$$

hence the matrix k is orthogonal, and $g = kp$.

(b) *Unicity.* Let $g \in GL(n, \mathbb{R})$ and assume that

$$g = kp = k_1 p_1,$$

where k and p are the matrices we considered in (a), and $k_1 \in O(n)$, $p_1 \in \mathcal{P}_n$. Let us show that $k_1 = k$, $p_1 = p$. Consider the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A = g^T g$, and let f be a polynomial in one variable such that

$$f(\lambda_i) = \sqrt{\lambda_i}, \quad i = 1, \ldots, n.$$

Then $p = f(A)$ and, since $p_1^2 = A$,

$$Ap_1 = p_1^3 = p_1 A,$$

therefore A and p_1 commute. It follows that $p = f(A)$ and p_1 commute and

$$k_1^{-1} k = p_1 p^{-1}.$$

The matrix $k_1 k^{-1}$, the product of two orthogonal matrices, is orthogonal. In general the product of two symmetric matrices A and B is not symmetric. However, if A and B commute, then the product AB is symmetric. One can diagonalise simultaneously the matrices A and B: there exists $h \in O(n)$ such that

$$A = h \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} h^{-1}, \quad B = h \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} h^{-1},$$

and

$$AB = h \begin{pmatrix} \lambda_1 \mu_1 & & \\ & \ddots & \\ & & \lambda_n \mu_n \end{pmatrix} h^{-1}.$$

Hence, if A and B are positive definite symmetric matrices, then the product AB is a positive definite symmetric matrix as well. Therefore, since the symmetric matrices p and p_1 commute and are positive definite, the matrix $p_1 p^{-1}$ is symmetric and positive definite. It follows that $k = k_1$, $p = p_1$ since

$$O(n) \cap \mathcal{P}_n = \{I\}.$$

In fact, assume that $g \in O(n) \cap \mathcal{P}_n$. Being orthogonal and symmetric, the matrix g satisfies $g = g^{-1}$. Its eigenvalues are then equal to ± 1. But since g is positive definite, its eigenvalues are all equal to 1, and $g = I$.

(c) *Continuity.* Clearly the map

$$O(n) \times \mathcal{P}_n \to GL(n, \mathbb{R}),$$
$$(k, p) \mapsto g = kp,$$

is continuous. To show that the inverse map is continuous, let us consider a convergent sequence (g_m) in $GL(n, \mathbb{R})$,

$$\lim_{m\to\infty} g_m = g.$$

Decompose each matrix g_m as $g_m = k_m p_m$. Let us show that $k_m \to k$ and $p_m \to p$, with $g = kp$. Since the group $O(n)$ is compact it is possible to extract from the sequence (k_m) a convergent subsequence (k_{m_j}),

$$\lim_{j\to\infty} k_{m_j} = k_0.$$

The sequence $(p_{m_j}) = (k_{m_j}^{-1} g_{m_j})$ also converges, with limit $p_0 = k_0^{-1} g$. Since it is the limit of positive definite symmetric matrices, the matrix p_0 is symmetric and semi-positive definite. Since g is invertible, p_0 is invertible too, hence $p_0 \in \mathcal{P}_n$, and

$$g = k_0 p_0.$$

By the uniqueness of the polar decomposition, $k_0 = k$, and k is the only accumulation point of the sequence (k_m), therefore (k_m) is a convergent sequence with limit k, and (p_m) converges to p. □

By diagonalising the matrix p in an orthogonal basis one obtains the following corollary.

Corollary 1.4.2 *Every element g in $GL(n, \mathbb{R})$ decomposes as*

$$g = k_1 d k_2,$$

with $k_1, k_2 \in O(n)$, and d is a diagonal matrix whose diagonal entries are strictly positive.

Note that the decomposition is not unique.

Let $GL(n, \mathbb{R})_+$ denote the subgroup of $GL(n, \mathbb{R})$ of matrices with positive determinant. Every element g in $GL(n, \mathbb{R})_+$ decomposes as

$$g = kp,$$

with $k \in SO(n)$, $p \in \mathcal{P}_n$, and also

$$g = k_1 d k_2,$$

with $k_1, k_2 \in SO(n)$, and d is a diagonal matrix with positive diagonal entries.