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Allan M. Sinclair and Roger R. Smith
Excerpt
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Chapter 1

General introduction

These notes are an introduction to some of the theory of finite von Neumann algebras and their von Neumann subalgebras, with the emphasis on maximal abelian self-adjoint subalgebras (usually abbreviated *masas*). Assuming basic von Neumann algebra theory, the notes are fairly detailed in covering the basic construction, perturbations of von Neumann subalgebras, general results on masas and detailed ones on singular masas in II_1 factors. Due to the large volume of research on finite von Neumann algebras and their masas the authors have been forced to be selective of the topics included. Nevertheless, a substantial body of recent research has been covered.

Each chapter of the book has its own introduction, so the overview of the contents below will be quite brief. We have also included a discussion of a few important results which have been omitted from the body of the text. In each case, we felt that the amount of background required for a reasonably self-contained account was simply too much for a book of this kind.

We have tried to make the material accessible to graduate students who have some familiarity with von Neumann algebras at the level of a first course in the subject. The early chapters review some of this, but are best read by the beginner with one of the standard texts, [104, 105, 187], to hand to fill in any gaps.

1.1 Synopsis

The book falls naturally into five parts. The first of these comprises Chapters 2, 3, 4, 5, 6 and 8 in which we lay out some of the foundations of the subject. The papers of Murray and von Neumann [116, 117, 202, 118] introduced the subject of von Neumann algebras (then called *Rings of Operators*) and are still influential today. The finite algebras are, roughly speaking, those that admit a faithful normal tracial state, and are closest in spirit to the matrix algebras, which are particular examples. Murray and von Neumann paid particular attention to the finite algebras, and established the close connection to the theory of discrete groups that persists today. They also introduced the crossed product construction from a group acting by automorphisms of a von Neumann algebra, notably an abelian one. In this way masas appear naturally in von Neumann algebras. In principle, masas can always be constructed by using Zorn's lemma, but this is rarely enlightening. The important examples always make their appearance from some auxiliary algebraic structure, for example maximal abelian subgroups of groups.

Chapter 2 classifies the masas in $B(H)$, the algebra of bounded operators on a separable Hilbert space. Although our focus is on II_1 factors, these cannot be studied in isolation, and various constructions will produce algebras of types I and II_∞ (but no type III factors will appear in these notes).

Chapter 3 gives an overview of the basic theory of finite von Neumann algebras and of the standard constructions of tensor products and crossed products. Since there is such a close connection to discrete groups, examples of masas arising from groups are presented, and the precursor to the conditional expectations are discussed at the level of group algebras where they are very easy to understand. An important characterisation of diffuse abelian algebras is given, and the chapter ends with a brief discussion of hyperfiniteness. The fundamental work of Connes, [36], on this topic is summarised without proofs.

The following chapter is devoted to the basic construction. This is an algebra $\langle N, e_B\rangle$ which arises from a von Neumann subalgebra B of a finite factor N. It is of fundamental importance in the theory of subfactors and also in perturbation theory. A detailed exposition of its properties is given, including the construction of its canonical semifinite trace (see also Appendix C for a different approach to this construction). Some simple examples are included.

Chapters 5, 6 and 8 deal with various technical issues. The first of these concerns the basic operators of von Neumann algebra theory—the unitaries, projections and partial isometries. Various approximation results are proved, and several important $\|\cdot\|_2$-norm estimates are given, all to be used subsequently. The next chapter continues in this spirit, and discusses various technical issues concerning normalising unitaries as well as orthogonality in von Neumann algebras. The background material is rounded out in Chapter 8 by presenting some estimates for operators in type I_∞ von Neumann algebras. We have avoided any discussion of direct integrals in these notes, but the material of Chapter 8 is essentially this topic in an embryonic form.

Chapter 7 introduces the Pukánszky invariant of a masa. At one level, all masas in separable II_1 factors are the same since all are isomorphic to $L^\infty[0,1]$. However, this ignores the relationship between a masa A and its containing factor N, and the invariant $\mathrm{Puk}(A)$ addresses this. The masa A and its reflection JAJ in the commutant combine to generate an abelian algebra whose commutant restricted to $L^2(N)\ominus L^2(A)$ decomposes as a direct sum of type I_n von Neumann algebras for n in the range $\{1,2,\ldots,\infty\}$. Those integers that appear then constitute $\mathrm{Puk}(A)$. There is a discussion of this invariant in the context of group factors where everything can be related to the group structure, allowing many examples to be presented. An alternative approach, based on important work of Tauer, [190], from the 1960s, has enabled White, [207], to show that all possible values of Puk(A) can be realised in certain factors including the hyperfinite one. Space considerations have forced us to omit this work.

The third group of chapters concerns perturbations of masas. This topic is split between Chapters 9 and 10, the first dealing with basic theory and second with extensions to general subalgebras. The problem is to consider two subalgebras A and B of a factor N which are close in an appropriate sense and then to look for a partial isometry w such that $wAw^* \subseteq B$. This creates a spatial isomorphism between compressions of the two algebras. This theory has played a decisive role in the resolution of some old questions in von Neumann algebra theory. We expand on this below.

Chapters 11–16 present various special aspects of masas. The focus of Chapter 11 is the theory of singular masas and we include a discussion of the Laplacian masa in free group factors. Chapter 12 is devoted to the construction of singular and semiregular masas in all finite factors. Chapter 13 explores the topic of Cartan masas, which is closely connected to the theory of hyperfinite subfactors, and there is also a discussion of property Γ and its relationship to masas. Maximally injective masas and subfactors are presented in Chapter 14, and the subsequent chapter looks at non-separable factors which can arise from ultrapowers. The last chapter presents some recent work of Shen [171] on singly generated algebras, a subject which relies heavily on the theory of masas.

The book concludes with three appendices. The first develops the theory of ultrapowers and includes some further material on property Γ. The second discusses the basic theory of unbounded operators. These types of operators appear in the perturbations of Chapters 9 and 10, so this appendix covers just that part of the theory which is used in these applications. The final appendix gives a second approach to the existence of the trace in the basic construction, first presented in Chapter 4.

1.2 Further results

There are three major results about masas which we have omitted from this book, due to the amount of background material that would have been needed to give a reasonably self-contained account of them.

The first of these is a uniqueness result for Cartan masas in the hyperfinite factor R. No automorphism of R can satisfy $\phi(A) = B$ when A and B are masas with distinct Pukánszky invariants, and such pairs do occur. However, the result of Connes, Feldman and Weiss, [40], for Cartan masas is as follows:

Theorem 1.2.1. *Let A and B be Cartan masas in the hyperfinite* II_1 *factor R. Then there exists an automorphism ϕ of R such that $\phi(A) = B$.*

The original proof in [40] is in the context of ergodic theory. A more operator-theoretic proof is presented in [141].

The second theorem answers an old question concerning the existence of Cartan masas in finite factors. The theory of free probability, [200], was developed as part of an investigation of the free group factors. In [199], this was used by Voiculescu to obtain

Theorem 1.2.2. *There do not exist Cartan masas in the free group factors* $L(\mathbb{F}_n)$, $2 \leq n \leq \infty$.

This settled the existence question for the standard types of masas since it had been shown earlier by Popa, [139, 140], that singular and semiregular masas exist in all separable II_1 factors (see Chapter 12).

The third problem that we wish to mention concerns the fundamental group, which is a multiplicative subgroup of $\mathbb{R}^+$ associated to any II_1 factor. The question is whether any subgroup can be the fundamental group of a II_1 factor. There had been some partial results that are discussed in Section 16.4, but the definitive answer was obtained by Popa [149]:

Theorem 1.2.3. *If G is a subgroup of $\mathbb{R}^+$, then there is a II_1 factor N whose fundamental group in G. If G is countable, then N may be taken to be separable.*

The proof depends on Gabariau's work in ergodic theory [70], as well as the perturbation results on masas discussed in Chapters 9 and 10.

Many mathematicians have made important contributions to the theory of masas, but amongst these two names stand out: Jacques Dixmier and Sorin Popa. This early work on masas in the 1950s is due to Dixmier, who did much to establish this topic as a viable field of study. Many of the later developments, in the periods 1980–1985 and from 2000 to the present, are due to Popa and several of our chapters are drawn substantially from his work. Without the fundamental contributions of these two researchers, this book could not have been written.

Chapter 2

Masas in $B(H)$

2.1 Introduction

Our main objective in this chapter is to describe the maximal abelian self-adjoint subalgebras (masas) of $B(H)$ in Section 2.3. To avoid technicalities concerning cardinalities, we will only discuss the case when H is separable. There are two basic types of masas, discrete and diffuse. For the first, fix an orthonormal basis $\{\xi_n\}_{n=1}^{\infty}$ for H and let p_n be the rank one projection onto $\mathbb{C}\xi_n$, $n \geq 1$. Then A, the von Neumann algebra generated by these projections, is a masa, and has many minimal projections. For the second type, let $L^\infty[0,1]$ act on $L^2[0,1]$ as multiplication operators. This is a masa, established below, but in contrast to the first type, it has no minimal projections. Up to unitary equivalence, each masa in $B(H)$ will be a direct sum of the two types.

In Section 2.4, we discuss masas in type I_n von Neumann algebras, where $n \in \mathbb{N}$ is arbitrary. These algebras have the form $A \otimes \mathbb{M}_n$ for some abelian von Neumann algebra A, where $\mathbb{M}_n$ denotes the algebra of scalar $n \times n$ matrices. Each of these algebras contains an obvious diagonal masa $A \otimes \mathbb{D}_n$, where $\mathbb{D}_n$ is the algebra of diagonal $n \times n$ matrices, and Theorem 2.4.3 establishes that this is the only masa up to unitary equivalence. We then conclude the chapter by introducing abelian projections and proving the useful result that they occur in all masas in finite type I von Neumann algebras.

Before we embark on the study of masas, we recall in Section 2.2 some of the standard theorems of von Neumann algebra theory. The ones chosen are those which will be used many times in the succeeding pages of this book.

2.2 Standard theorems

There are many important theorems in operator algebras, but the four that we recall here are of particular relevance to the topics of these notes. They can be found in all of the textbooks in this field, and are restated here for the reader's convenience (see, for example, the books by Dixmier [48, 49], Kadison and Ringrose [104, 105], Pedersen [131], Sakai [167] and Takesaki [187]). All are valid without restriction on the cardinality of the Hilbert space.

If a von Neumann algebra N is represented on a Hilbert space H, then a net (x_α) from N converges strongly to $x \in N$ if $\lim_\alpha \|x_\alpha \eta - x\eta\|_2 = 0$ for all vectors $\eta \in H$. If, in addition, $\lim_\alpha x_\alpha^* = x^*$ strongly, then we say that $x_\alpha \to x$ *-strongly. These notions are distinct: if v denotes the adjoint of the unilateral shift operator on $\ell^2(\mathbb{N})$, then $\lim_{n\to\infty} v^n = 0$ strongly, while this is not so for powers of v^*. Basic functional analysis shows that weak and strong closures of convex sets coincide, giving a choice of how to state these results.

The first one, due to von Neumann, [201], describes the strong closure of a *-algebra of operators in purely algebraic terms.

Theorem 2.2.1 (**The double commutant theorem).** *Let A be a $*$-algebra of operators on a Hilbert space H and suppose that 1 is in the strong (or weak) closure of A. Then the strong and weak closures of A are both equal to the double commutant A''. In particular, A is a von Neumann algebra if and only if $A = A''$.*

The next theorem, due to Dixmier, [46], is a type of averaging result, and such techniques will appear frequently in these notes, taking various forms. We let Z denote the centre of a von Neumann algebra N, and the closure of the convex set below is taken in the norm topology. Another way of describing the theorem is to say that appropriately chosen convex combinations of unitary conjugates of any fixed element approach the centre arbitrarily closely. The simplicity of finite factors is one consequence of this result (see Theorem A.3.2).

Theorem 2.2.2 (**The Dixmier approximation theorem).** *Let N be a von Neumann algebra with unitary group $\mathcal{U}(N)$. For each $x \in N$,*

$$Z \cap \overline{\text{conv}}\{uxu^* : u \in \mathcal{U}(N)\} \neq \emptyset.$$

The third theorem is due to Kaplansky [106]. The most important feature for us is the norm estimate of the first part, since it is quite possible to have nets which are unbounded in norm, but nevertheless converge strongly. Our statement of the result combines the versions of [105, 187].

Theorem 2.2.3 (**The Kaplansky density theorem).** *Let $N \subseteq B(H)$ be a von Neumann algebra and let A be a strongly dense *-subalgebra, not assumed to be unital.*

(i) *If $x \in N$, then there exists a net (x_α) from A converging *-strongly to x and satisfying $\|x_\alpha\| \leq \|x\|$ for all α.*

(ii) *If $x \in N$ is self-adjoint then the net in (i) may be chosen with the additional property that each x_α is self-adjoint.*

(iii) *If $u \in N$ is a unitary and A is a unital C^*-algebra, then there is a net (u_α) of unitaries from A converging *-strongly to u.*

The last of these theorems is due to Tomita and may be found in [105, Theorem 11.2.16]. It is a consequence of another of Tomita's theorems, [105, Theorem 9.2.9], that has played a fundamental role in the modern development of certain aspects of the theory. The simplicity of its statement is in contrast to the difficulty of its proof.

Theorem 2.2.4 (Tomita's commutant theorem). *Let $M \subseteq B(H)$ and $N \subseteq B(K)$ be von Neumann algebras. Then*

$$(M\overline{\otimes}N)' = M'\overline{\otimes}N'.$$

Early in the development of the subject, von Neumann introduced direct integrals [203] which allowed him to decompose a von Neumann algebra on a separable Hilbert space into a direct integral over the centre of factors (those algebras with trivial centre). This focused attention on factors, the prevailing view being that direct integral theory would extend results for factors to separably acting von Neumann algebras, which is largely correct. Murray and von Neumann introduced the type classification of factors in their seminal series of papers [116, 117, 118, 202]. There are algebras of types I, II_1, II_∞ and III. The reader will find the original definitions in [105], which also contains theorems making these equivalent to the following descriptions. Type I breaks down further into I_n, $n \geq 1$, and type I_∞, and these correspond respectively to the algebras $\mathbb{M}_n$ of $n \times n$ matrices over $\mathbb{C}$, and to the algebras $B(H)$ of bounded operators on infinite dimensional Hilbert spaces H, one for each cardinality of H. The II_1 factors are those which are infinite dimensional and admit a finite trace, and are often called finite factors. The II_∞ factors are those which arise from the tensor product of a type II_1 with a type I_∞, while any factor which does not fall into the classes already described is called type III. The original definitions also cover algebras with nontrivial centre and the direct integral theory works well here: a separably acting von Neumann algebra is of type α precisely when the factors in its direct integral decomposition are all of type α ($\alpha \in \{\mathrm{I}, \mathrm{II}_1, \mathrm{II}_\infty, \mathrm{III}\}$). The type III algebras play no further role in these notes, which mainly concern those of type II_1. However, various constructions lead to algebras of types I and II_∞. As we will see, the trace is fundamental for finite algebras, and we lose this when we move to the other two types. Fortunately, both have densely defined semifinite traces, and considerable use will be made of this.

All von Neumann algebras have identity elements, which we usually denote by 1. We will often need to consider containments $B \subseteq N$ of von Neumann algebras, and we adopt the following convention. We always suppose that the identities of N and B are equal unless **either** it is explicitly stated to the contrary, **or** it is clearly not the case from the context (e.g. $B = pNp$ for some projection $p \in N$).

2.3 Masas

The material in this section owes much to the presentation of these results to be found in [104, Chapter 5].

Recall that a vector ξ is *cyclic* for a von Neumann algebra $N \subseteq B(H)$ if the subspace $N\xi$ is norm dense in H. We say that ξ is *separating* for N if $x\xi = 0$ implies that $x = 0$ when $x \in N$. The first lemma gives a useful relationship between cyclic and separating vectors.

Lemma 2.3.1. *Let $N \subseteq B(H)$ be a von Neumann algebra. Then $\xi \in H$ is cyclic for N if and only if ξ is separating for N'.*

Proof. Suppose that ξ is cyclic for N, and let $x' \in N'$ be such that $x'\xi = 0$. Then

$$x'x\xi = xx'\xi = 0, \qquad x \in N, \tag{2.3.1}$$

and so $x' = 0$ since $N\xi$ is dense in H. Thus ξ is separating for N'.

Conversely, suppose that ξ is separating for N'. Let $p \in N'$ be the projection onto the norm closed span of $N\xi$. Then $p\xi = \xi$, so $(1-p)\xi = 0$. Since ξ is separating, $p = 1$, which says that ξ is cyclic for N. □

Both examples of discrete and diffuse masas above have cyclic vectors. In the first case $\xi = \sum_{n=1}^{\infty} \xi_n/2^n$ is cyclic, while in the second case the constant function 1 is cyclic for $L^\infty[0,1]$. The situation would change if we allowed non-separable Hilbert spaces. If S is an uncountable set then $\ell^\infty(S)$ is a masa when acting by multiplication on $\ell^2(S)$. Any vector in $\ell^2(S)$ has only a countable number of non-zero entries, and thus no cyclic vector can exist. The obstructions, of course, are the cardinality S and the resulting dimension of $\ell^2(S)$.

Lemma 2.3.2. *Let $A \subseteq B(H)$ be a masa, where H is a separable Hilbert space. Then there is a vector $\xi \in H$ which is both cyclic and separating for A.*

Proof. By Zorn's lemma, choose a maximal set $\{\xi_n\}_{n=1}^{\infty}$ of non-zero vectors such that the subspaces $\overline{A\xi_n}$, $n \geq 1$, are pairwise orthogonal. The separability of H allows us to enumerate this set. If $\eta \neq 0$ was orthogonal to all of these subspaces then $\overline{A\eta}$ would be orthogonal to each $\overline{A\xi_n}$, contradicting maximality. Thus $H = \bigoplus_{n=1}^{\infty} \overline{A\xi_n}$. Let p_n be the projection onto $\overline{A\xi_n}$, $n \geq 1$. Then $p_n \in A' = A$. Let $\xi = \sum_{n=1}^{\infty} \xi_n/(2^n\|\xi_n\|_2) \in H$. Then $\xi_n = (2^n\|\xi_n\|_2 p_n)(\xi)$, so $\xi_n \in \overline{A\xi}$, $n \geq 1$. It then follows that $\overline{A\xi_n} \subseteq \overline{A\xi}$, $n \geq 1$, and so ξ is cyclic for A. By Lemma 2.3.1, ξ is separating for A', which equals A. □

We now establish a converse to this result, by showing that any abelian von Neumann algebra on a separable Hilbert space which has a cyclic vector must be a masa. This then proves that the example $L^\infty[0,1]$ above is indeed a masa on $L^2[0,1]$. We will require a preliminary lemma concerning tracial vectors. A vector $\xi \in H$ is said to be *tracial* for a von Neumann algebra $N \subseteq B(H)$ if

$$\langle xy\xi, \xi\rangle = \langle yx\xi, \xi\rangle, \qquad x, y \in N. \tag{2.3.2}$$

Note that if N is abelian then any vector is tracial for N.

Lemma 2.3.3. *Let $N \subseteq B(H)$ be a von Neumann algebra with a tracial cyclic and separating vector ξ. Then there is a conjugate linear isometry $J\colon\ H \to H$ with the following properties:*

(i) *$t \mapsto Jt^*J$, $t \in B(H)$, defines an anti-isomorphism $\pi\colon\ B(H) \to B(H)$;*

(ii) *π maps N onto N' and N' onto N.*

Proof. (i) For each $x \in N$,

$$\|x\xi\|_2^2 = \langle x\xi, x\xi\rangle = \langle x^*x\xi, \xi\rangle = \langle xx^*\xi, \xi\rangle = \|x^*\xi\|_2^2, \tag{2.3.3}$$

so we may define a conjugate linear isometry $J\colon\ N\xi \to N\xi$ by $J(x\xi) = x^*\xi$, $x \in N$. Since ξ is cyclic for N, this map extends to H, also denoted J. If we define $\pi\colon\ B(H) \to B(H)$ by $\pi(t) = Jt^*J$ then, for $s, t \in B(H)$,

$$\pi(st) = Jt^*s^*J = Jt^*JJs^*J = \pi(t)\pi(s), \tag{2.3.4}$$

using $J^2 = I$. This proves (i).
(ii) Now consider $x, y, z \in N$. We have

$$\begin{aligned}(x\pi(y) - \pi(y)x)z\xi &= xJy^*Jz\xi - Jy^*Jxz\xi\\ &= xJy^*z^*\xi - Jy^*z^*x^*\xi\\ &= xzy\xi - xzy\xi = 0.\end{aligned} \tag{2.3.5}$$

Letting z vary over N, we conclude that $\pi(y)$ commutes with all $x \in N$, so $\pi(y) \in N'$. Thus π maps N to N'.

Let $y \in N'$ be self-adjoint and choose a sequence $\{x_n\}_{n=1}^\infty \in N$ such that $y\xi = \lim_{n\to\infty} x_n\xi$, possible because ξ is cyclic for N. Then $\{x_n^*\xi\}_{n=1}^\infty$ is Cauchy, since $x_n^*\xi = Jx_n\xi$, and so this sequence converges to $\eta \in H$. For $z \in N$,

$$\begin{aligned}\langle y\xi - \eta, z\xi\rangle &= \lim_{n\to\infty} (\langle y\xi, z\xi\rangle - \langle x_n^*\xi, z\xi\rangle)\\ &= \lim_{n\to\infty} (\langle z^*\xi, y\xi\rangle - \langle z^*x_n^*\xi, \xi\rangle)\\ &= \lim_{n\to\infty} (\langle z^*\xi, y\xi\rangle - \langle x_n^*z^*\xi, \xi\rangle)\\ &= \lim_{n\to\infty} (\langle z^*\xi, y\xi\rangle - \langle z^*\xi, x_n\xi\rangle) = 0,\end{aligned} \tag{2.3.6}$$

where we have used the tracial property of ξ and the fact that y is self-adjoint and commutes with z^*. Letting z vary, we conclude that $y\xi = \eta$, and so $y\xi$ is the limit of the sequence $\{((x_n + x_n^*)/2)\xi\}_{n=1}^\infty$. Replacing x_n by $(x_n + x_n^*)/2$, we may assume that x_n is self-adjoint. Then

$$Jy\xi = \lim_{n\to\infty} Jx_n\xi = \lim_{n\to\infty} x_n\xi = y\xi. \tag{2.3.7}$$

A general element $y \in N'$ may be written as $y = y_1 + iy_2$ with y_1 and y_2 self-adjoint in N'. From (2.3.7), we obtain $Jy\xi = y^*\xi$. By Lemma 2.3.1, ξ is also cyclic and separating for N', so we may repeat the argument of (2.3.5), this time with $x, y, z \in N'$, to conclude that π maps N' to N. Since $\pi^2 = I$, it follows easily that π defines an anti-isomorphismof N onto N', proving (ii). □

Theorem 2.3.4. *Let $A \subseteq B(H)$ be an abelian von Neumann algebra on a separable Hilbert space H. Then A is a masa if and only if A has a cyclic vector.*

Proof. We have already shown in Lemma 2.3.2 that if A is a masa then it has a cyclic vector. To show the converse, suppose that A has a cyclic vector ξ. Since A is abelian, we have that $A \subseteq A'$, and ξ is also cyclic for A'. By Lemma 2.3.1, ξ is separating for A, and it is of course tracial. By Lemma 2.3.3, there is a surjective anti-isomorphism $\pi\colon\ A \to A'$, showing that A' is also abelian. Thus $A' \subseteq (A')' = A$, proving that $A = A'$. Then A is a masa. □

Our next objective is to give a complete description of masas in $B(H)$. We have already met two types: $L^\infty[0,1]$ acting on $L^2[0,1]$ and the masas of diagonal operators relative to given orthonormal bases. A general masa will be a direct sum of the two types (where either type may be missing), and the diagonal masas are characterised by the cardinality of the orthonormal bases. We call a masa *diffuse* (or *continuous*) if it has no minimal non-zero projections. If it is generated by its minimal projections then we refer to it as *discrete*. The following lemma is valid for all separably acting von Neumann algebras and so we prove it in full generality, although we only apply it to masas. It will be useful in describing diffuse masas.

Lemma 2.3.5. *Let H be a separable Hilbert space and let $N \subseteq B(H)$ be a von Neumann algebra. Then there is a sequence $\{p_n\}_{n=1}^\infty$ of projections in N which generates this von Neumann algebra.*

Proof. The spectral theorem allows us to approximate a given self-adjoint operator by a finite linear combination of its spectral projections, and each operator is in the span of two self-adjoint ones. Thus we may reduce to showing that N is generated by a countable set of elements.

Fix a countable dense set of vectors $\{\xi_i\}_{i=1}^\infty$ in H. Let n be a fixed integer, and let

$$S_n = \{(x\xi_1, \ldots, x\xi_n)^T\colon\ x \in N,\ \|x\| \le 1\} \subseteq H^n.$$

Then S_n is separable, and so there is a countable subset F_n of the unit ball of N such that $\{(x\xi_1, \ldots, x\xi_n)^T\colon\ x \in F_n\}$ is norm dense in S_n. A simple approximation argument then shows that $\bigcup_{n=1}^\infty F_n$ is a countable strongly dense subset of N, and thus generates the von Neumann algebra. □

Lemma 2.3.6. *Let A be a diffuse masa in $B(H)$, where H is a separable Hilbert space. Then A is unitarily equivalent to the masa $L^\infty[0,1]$ in $B(L^2[0,1])$.*

Proof. For each $\lambda \in [0,1]$, define the projection $f_\lambda \in L^\infty[0,1]$ to be $\chi_{[0,\lambda]}$. Since the constant function $1 \in L^2[0,1]$ is a cyclic vector for the von Neumann algebra B generated by these projections, Theorem 2.3.4 shows that B is a masa, and thus coincides with $L^\infty[0,1]$. We note that the set of f_λ's is totally ordered and that $f_\lambda f_\mu = f_{\min\{\lambda,\mu\}}$. We will construct a set of projections indexed by [0,1] inside A with similar properties, from which we will obtain the implementing unitary.