

Contents

0	Introduction	<i>page</i> 1
I	The Topology of Algebraic Varieties	17
1	The Lefschetz Theorem on Hyperplane Sections	19
1.1	Morse theory	20
1.1.1	Morse's lemma	20
1.1.2	Local study of the level set	23
1.1.3	Globalisation	27
1.2	Application to affine varieties	28
1.2.1	Index of the square of the distance function	28
1.2.2	Lefschetz theorem on hyperplane sections	31
1.2.3	Applications	34
1.3	Vanishing theorems and Lefschetz' theorem	36
	Exercises	39
2	Lefschetz Pencils	41
2.1	Lefschetz pencils	42
2.1.1	Existence	42
2.1.2	The holomorphic Morse lemma	46
2.2	Lefschetz degeneration	47
2.2.1	Vanishing spheres	47
2.2.2	An application of Morse theory	48
2.3	Application to Lefschetz pencils	53
2.3.1	Blowup of the base locus	53
2.3.2	The Lefschetz theorem	54
2.3.3	Vanishing cohomology and primitive cohomology	57
2.3.4	Cones over vanishing cycles	60
	Exercises	62

3	Monodromy	67
3.1	The monodromy action	69
3.1.1	Local systems and representations of π_1	69
3.1.2	Local systems associated to a fibration	73
3.1.3	Monodromy and variation of Hodge structure	74
3.2	The case of Lefschetz pencils	77
3.2.1	The Picard–Lefschetz formula	77
3.2.2	Zariski’s theorem	85
3.2.3	Irreducibility of the monodromy action	87
3.3	Application: the Noether–Lefschetz theorem	89
3.3.1	The Noether–Lefschetz locus	89
3.3.2	The Noether–Lefschetz theorem	93
	Exercises	94
4	The Leray Spectral Sequence	98
4.1	Definition of the spectral sequence	100
4.1.1	The hypercohomology spectral sequence	100
4.1.2	Spectral sequence of a composed functor	107
4.1.3	The Leray spectral sequence	109
4.2	Deligne’s theorem	113
4.2.1	The cup-product and spectral sequences	113
4.2.2	The relative Lefschetz decomposition	115
4.2.3	Degeneration of the spectral sequence	117
4.3	The invariant cycles theorem	118
4.3.1	Application of the degeneracy of the Leray–spectral sequence	118
4.3.2	Some background on mixed Hodge theory	119
4.3.3	The global invariant cycles theorem	123
	Exercises	124
II	Variations of Hodge Structure	127
5	Transversality and Applications	129
5.1	Complexes associated to IVHS	130
5.1.1	The de Rham complex of a flat bundle	130
5.1.2	Transversality	133
5.1.3	Construction of the complexes $K_{l,r}$	137
5.2	The holomorphic Leray spectral sequence	138
5.2.1	The Leray filtration on Ω_X^p and the complexes $\mathcal{K}_{p,q}$	138
5.2.2	Infinitesimal invariants	141
5.3	Local study of Hodge loci	143
5.3.1	General properties	143
5.3.2	Infinitesimal study	146

<i>Contents</i>		vii
5.3.3	The Noether–Lefschetz locus	148
5.3.4	A density criterion	151
	Exercises	153
6	Hodge Filtration of Hypersurfaces	156
6.1	Filtration by the order of the pole	158
6.1.1	Logarithmic complexes	158
6.1.2	Hodge filtration and filtration by the order of the pole	160
6.1.3	The case of hypersurfaces of \mathbb{P}^n	163
6.2	IVHS of hypersurfaces	167
6.2.1	Computation of $\bar{\nabla}$	167
6.2.2	Macaulay’s theorem	171
6.2.3	The symmetriser lemma	175
6.3	First applications	177
6.3.1	Hodge loci for families of hypersurfaces	177
6.3.2	The generic Torelli theorem	179
	Exercises	184
7	Normal Functions and Infinitesimal Invariants	188
7.1	The Jacobian fibration	189
7.1.1	Holomorphic structure	189
7.1.2	Normal functions	191
7.1.3	Infinitesimal invariants	192
7.2	The Abel–Jacobi map	193
7.2.1	General properties	193
7.2.2	Geometric interpretation of the infinitesimal invariant	197
7.3	The case of hypersurfaces of high degree in \mathbb{P}^n	205
7.3.1	Application of the symmetriser lemma	205
7.3.2	Generic triviality of the Abel–Jacobi map	207
	Exercises	212
8	Nori’s Work	215
8.1	The connectivity theorem	217
8.1.1	Statement of the theorem	217
8.1.2	Algebraic translation	218
8.1.3	The case of hypersurfaces of projective space	223
8.2	Algebraic equivalence	228
8.2.1	General properties	228
8.2.2	The Hodge class of a normal function	229
8.2.3	Griffiths’ theorem	233

8.3	Application of the connectivity theorem	235
8.3.1	The Nori equivalence	235
8.3.2	Nori's theorem	237
	Exercises	240
III	Algebraic Cycles	243
9	Chow Groups	245
9.1	Construction	247
9.1.1	Rational equivalence	247
9.1.2	Functoriality: proper morphisms and flat morphisms	248
9.1.3	Localisation	254
9.2	Intersection and cycle classes	256
9.2.1	Intersection	256
9.2.2	Correspondences	259
9.2.3	Cycle classes	261
9.2.4	Compatibilities	263
9.3	Examples	269
9.3.1	Chow groups of curves	269
9.3.2	Chow groups of projective bundles	269
9.3.3	Chow groups of blowups	271
9.3.4	Chow groups of hypersurfaces of small degree	273
	Exercises	275
10	Mumford's Theorem and its Generalisations	278
10.1	Varieties with representable CH_0	280
10.1.1	Representability	280
10.1.2	Roitman's theorem	284
10.1.3	Statement of Mumford's theorem	289
10.2	The Bloch–Srinivas construction	291
10.2.1	Decomposition of the diagonal	291
10.2.2	Proof of Mumford's theorem	294
10.2.3	Other applications	298
10.3	Generalisation	301
10.3.1	Generalised decomposition of the diagonal	301
10.3.2	An application	303
	Exercises	304
11	The Bloch Conjecture and its Generalisations	307
11.1	Surfaces with $p_g = 0$	308
11.1.1	Statement of the conjecture	308
11.1.2	Classification	310

Contents

ix

11.1.3	Bloch's conjecture for surfaces which are not of general type	313
11.1.4	Godeaux surfaces	315
11.2	Filtrations on Chow groups	322
11.2.1	The generalised Bloch conjecture	322
11.2.2	Conjectural filtration on the Chow groups	324
11.2.3	The Saito filtration	327
11.3	The case of abelian varieties	328
11.3.1	The Pontryagin product	328
11.3.2	Results of Bloch	329
11.3.3	Fourier transform	336
11.3.4	Results of Beauville	339
	Exercises	340
	<i>References</i>	343
	<i>Index</i>	348