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# HODGE THEORY AND COMPLEX ALGEBRAIC GEOMETRY II

CLAIRE VOISIN

*Institut de Mathématiques de Jussieu*

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