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Harmonic Analysis on Compact Symmetric Spaces: the Legacy of Elie Cartan and Hermann Weyl

Roe Goodman

Department of Mathematics
Rutgers, The State University of New Jersey

1 Introduction

In his lecture *Relativity theory as a stimulus in mathematical research* [Wey4], Hermann Weyl says that "Frobenius and Issai Schur's spadework on finite and compact groups and Cartan's early work on semi-simple Lie groups and their representations had nothing to do with it [relativity theory]. But for myself I can say that the wish to understand what really is the mathematical substance behind the formal apparatus of relativity theory led me to the study of representations and invariants of groups, and my experience in this regard is probably not unique."

Weyl's first encounter with Lie groups and representation theory as a tool to understand relativity theory occurred in connection with the Helmholtz-Lie space problem and the problem of decomposing the tensor product $\otimes^k \mathbb{C}^n$ under the mutually commuting actions of the general linear group $\mathrm{GL}(n, \mathbb{C})$ (on each copy of $\mathbb{C}^n$) and the symmetric group $\mathfrak{S}_k$ (in permuting the k copies of $\mathbb{C}^n$).[1] He later described the tensor decomposition problem in general terms [Wey3] as "an epistemological principle basic for all theoretical science, that of projecting the actual upon the background of the possible." Mathematically, the issue was to find subspaces of tensor space that are invariant and irreducible under all transformations that commute with $\mathfrak{S}_k$. This had already been done by Frobenius and Schur around 1900, but apparently Weyl first became aware of these results in the early 1920's. The subspaces in question, which are the ranges of minimal projections in the group algebra of $\mathfrak{S}_k$, are exactly the irreducible (polynomial) representations of $\mathrm{GL}(n, \mathbb{C})$, and all irreducible representations arise this way for varying k by including multiplication by integral powers of $\det(g)$ in the action. It seems clear

1 see [Haw, §11.2-3]

from his correspondence with Schur at this time that these results were Weyl's starting point for his later work in representation theory and invariant theory.

Near the end of his monumental paper on representations of semisimple Lie groups [Wey1, Kap. IV, §4], Weyl considers the problem of constructing all the irreducible representations of a simply-connected simple Lie group G such as $\mathrm{SL}(n, \mathbb{C})$. This had been done on a case-by-case basis by Cartan [Car1], starting with the defining representations for the classical groups (or the adjoint representation for the exceptional groups) and building up a general irreducible representation by forming tensor products. By contrast, Weyl, following the example of Frobenius for finite groups, says that "the correct starting point for building representations does not lie in the adjoint group, but rather in the *regular representation*, which through its reduction yields *in one blow* all irreducible representations." He introduces the infinite-dimensional space $C(U)$ of all continuous functions on the compact real form U of G ($U = \mathrm{SU}(n)$ when $G = \mathrm{SL}(n, \mathbb{C})$) and the right translation representation of U on $C(U)$. He then obtains the irreducible representations of U and their characters by using the eigenspaces of compact integral operators given by left convolution with positive-definite functions in $C(U)$, in analogy with the decomposition of tensor spaces for $\mathrm{GL}(n, \mathbb{C})$ using elements of the group algebra of $\mathfrak{S}_k$. The details are spelled out in the famous Peter–Weyl paper [Pe-We], which proves that the normalized matrix entries of the irreducible unitary representations of U furnish an orthonormal basis for $L^2(U)$, and that every continuous function on U is a uniform limit of linear combinations of these matrix entries.

In the introduction to [Car2], É. Cartan says that his paper was inspired by the paper of Peter and Weyl, but he points out that for a compact Lie group their use of integral equations "gives a transcendental solution to a problem of an algebraic nature" (namely, the completeness of the set of finite-dimensional irreducible representations of the group). Cartan's goal is "to give an algebraic solution to a problem of a transcendental nature, more general than that treated by Weyl." Namely, to find an explicit decomposition of the space of all L^2 functions on a homogeneous space into an orthogonal direct sum of group-invariant irreducible subspaces.

Cartan's paper [Car2] then stimulated Weyl [Wey2] to treat the same problem again and write "the systematic exposition by which I should like to replace the two papers Peter–Weyl [Pe-We] and Cartan [Car2]." In his characteristic style of finding the core of a problem through generalization, Weyl takes the finite-dimensional irreducible subspaces of functions (which he calls the *harmonic sets* by analogy with the case of spherical harmonics) on the compact homogeneous space X as his starting point.[2] Using the invariant measure on the homogenous space, he constructs integral operators that intertwine the representation of the compact group U on $C(X)$ with the left regular representation on $C(U)$.

In this paper we approach the Weyl–Cartan results by way of algebraic groups. The *finite* functions on a homogeneous space for a compact connected Lie group (that is, the functions whose translates span a finite-dimensional subspace) can be viewed as *regular* functions on the complexified group (a complex reductive algebraic group). Irreducible subspaces of functions under the action of the compact group correspond to irreducible subspaces of regular functions on the complex reductive group—this is Weyl's *unitarian trick*. We describe the algebraic group version of the Peter–Weyl decomposition and geometric criterion for simple spectrum of a homogeneous space (due to E. Vinberg and B. Kimelfeld). We present R. Richardson's algebraic group version of the Cartan embedding of a symmetric space, and the celebrated results of Cartan and S. Helgason concerning finite-dimensional spherical representations.

We then turn to more recent results of J.-L. Clerc [Cle] concerning the complexified Iwasawa decomposition and zonal spherical functions on a compact symmetric space, and S. Gindikin's construction ([Gin1], [Gin2], [Gin3]) of the *horospherical Cauchy–Radon transform*, which shows that compact symmetric spaces have canonical dual objects that are complex manifolds.

We make frequent citations to the extraordinary books of A. Borel [Bor] and T. Hawkins [Haw], which contain penetrating historical accounts of the contributions of Weyl and Cartan. Borel's book also describes the development of algebraic groups by C. Chevalley that is basic to our approach. For a survey of other developments in harmonic analysis on symmetric spaces from Cartan's paper to the mid 1980's see Helgason [Hel3]. Thanks go to the referee for pointing out some notational inconsistencies and making suggestions for improving the organization of this paper.

2 Weyl's emphasis on function spaces, rather than the underlying homogeneous space, is in the spirit of the recent development of *quantum groups*; his immediate purpose was to make his theory sufficiently general to include also J. von Neumann's theory of almost-periodic functions on groups, in which the functions determine a compactification of the underlying group.

2 Algebraic Group Version of Peter–Weyl Theorem

2.1 Isotypic Decomposition of $\mathcal{O}[X]$

The paper [Pe-We] of Peter and Weyl considers compact Lie groups U; because the group is compact left convolution with a continuous function is a compact operator. Hence such an operator, if self-adjoint, has finite-dimensional eigenspaces that are invariant under right translation by elements of U. The finiteness of the invariant measure on U also guarantees that every finite-dimensional representation of U carries a U-invariant positive-definite inner product, and hence is *completely reducible* (decomposes as the direct sum of irreducible representations).[3]

Turning from Weyl's transcendental methods to the more algebraic and geometric viewpoint preferred by Cartan, we recall that a subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is an *algebraic group* if it is the zero set of a collection of polynomials in the matrix entries. The *regular functions* $\mathcal{O}[G]$ are the restrictions to G of polynomials in matrix entries and $\det^{-1}$. In particular, G is a complex Lie group and the regular functions on G are holomorphic. A finite-dimensional complex representation (π, V) of G is *rational* if the matrix entries of the representation are regular functions on G. The group G is *reductive* if every rational representation is completely reducible.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. From the work of Cartan, Weyl, and Chevalley, one knows the following:

(1) There is a simply-connected complex linear algebraic group G with Lie algebra $\mathfrak{g}$.

(2) The finite-dimensional representations of $\mathfrak{g}$ correspond to rational representations of G.

(3) There is a real form $\mathfrak{u}$ of $\mathfrak{g}$ and a simply-connected compact Lie group $U \subset G$ with Lie algebra $\mathfrak{u}$.

(4) The finite-dimensional unitary representations of U extend uniquely to rational representations of G, and U-invariant subspaces correspond to G-invariant subspaces.[4]

(5) The irreducible rational representations of G are parameterized by the positive cone in a lattice of rank l (Cartan's theorem of the *highest weight*).[5]

3 This is the Hurwitz "trick" (*Kunstgriff*) that Weyl learned from I. Schur; see Hawkins [Haw, §12.2].

4 This is Weyl's *unitary trick*.

5 The first algebraic proofs of this that did not use case-by-case considerations were found by Chevalley and Harish-Chandra in 1948; see [Bor, Ch. VII, §3.6-7].

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The highest weight construction is carried out as follows: Fix a Borel subgroup $B = HN^+$ of G (a maximal connected solvable subgroup). Here $H \cong (\mathbb{C}^\times)^l$, with $l = \mathrm{rank}(G)$, is a maximal algebraic torus in G, and N^+ is the unipotent radical of B associated with a set of positive roots of H on $\mathfrak{g}$. Let $\bar{B} = HN^-$ be the opposite Borel subgroup. We can always arrange the embedding $G \subset \mathrm{GL}(n, \mathbb{C})$ so that H consists of the diagonal matrices in G, N^+ consists of the upper-triangular unipotent matrices in G, and N^- consists of the lower-triangular unipotent matrices in G. Let $\mathfrak{h}$ be the Lie algebra of H and $\Phi \subset \mathfrak{h}^*$ the roots of $\mathfrak{h}$ on $\mathfrak{g}$. Write $P(\Phi) \subset \mathfrak{h}^*$ for the *weight lattice* of H and $P_{++} \subset P(\Phi)$ for the *dominant weights*, relative to the system of positive roots determined by N^+. For $\lambda \in P(\Phi)$ we denote by $h \mapsto h^\lambda$ the corresponding character of H. It extends to a character of B by $(hn)^\lambda = h^\lambda$ for $h \in H$ and $n \in N^+$.

An irreducible rational representation (π, E) of G is then determined (up to equivalence) by its *highest weight*. The subspace E^{N^+} of N^+-fixed vectors in E is one-dimensional, and H acts on it by a character $h \mapsto h^\lambda$ where $\lambda \in P_{++}$. The subspace E^{N^-} of N^--fixed vectors in E is also one-dimensional, and H acts on it by the character $h \mapsto h^{-\lambda^*}$ where $\lambda^* = -w_0 \cdot \lambda$. Here w_0 is the element of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ that interchanges positive and negative roots.

For each $\lambda \in P_{++}$ we fix a model (π_λ, E_λ) for the irreducible rational representation with highest weight λ. Then $(\pi_{\lambda^*}, E_\lambda^*)$ is the contragredient representation. Fix a highest weight vector $e_\lambda \in E_\lambda$ and a lowest weight vector $f_{\lambda^*} \in E_\lambda^*$, normalized so that

$$\langle e_\lambda, f_{\lambda^*} \rangle = 1.$$

Here we are using $\langle v, v^* \rangle$ to denote the tautological duality pairing between a vector space and its dual (in particular, this pairing is complex linear in both arguments). For dealing with matrix entries as regular functions on the complex algebraic group G this is more convenient than using a U-invariant inner product on E_λ and identifying E_λ^* with E_λ via a conjugate-linear map.

Let X be an irreducible affine algebraic G space. Denote the regular functions on X by $\mathcal{O}[X]$. There is a representation ρ of G on $\mathcal{O}[X]$:

$$\rho(g)f(x) = f(g^{-1}x) \quad \text{for } f \in \mathcal{O}[X] \text{ and } g \in G.$$

Because the G-action is algebraic, $\mathrm{Span}\{\rho(G)f\}$ is a finite-dimensional rational G-module for $f \in \mathcal{O}[X]$. There is a tautological G-intertwining map

$$E_\lambda \otimes \mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) \to \mathcal{O}[X],$$

given by $v \otimes T \mapsto Tv$. For $\lambda \in P_{++}$ let

$$\mathcal{O}[X]^{N^+}(\lambda) = \{f \in \mathcal{O}[X] : \rho(hn)f = h^\lambda f \quad \text{for } h \in H \text{ and } n \in N^+\}. \tag{2.1}$$

The key point is that the choice of a highest weight vector e_λ gives an isomorphism

$$\mathrm{Hom}_G(E_\lambda, \mathcal{O}[X]) \cong \mathcal{O}[X]^{N^+}(\lambda). \tag{2.2}$$

Here a G-intertwining map T applied to the highest weight vector gives the function $\varphi = Te_\lambda \in \mathcal{O}[X]^{N^+}(\lambda)$, and conversely every such function φ defines a unique intertwining map T by this formula.[6] From (2.2) we see that the highest weights of the G-irreducible subspaces of $\mathcal{O}[X]$ comprise the set

$$\mathrm{Spec}(X) = \{\lambda \in P_{++} : \mathcal{O}[X]^{N^+}(\lambda) \neq 0\} \quad (\text{the } G \textit{ spectrum} \text{ of } X)$$

Using the isomorphism (2.2) and the reductivity of G, we obtain the decomposition of $\mathcal{O}[X]$ under the action of G, as follows:

Theorem 2.1 *The isotypic subspace of type (π_λ, E_λ) in $\mathcal{O}[X]$ is the linear span of the G-translates of $\mathcal{O}[X]^{N^+}(\lambda)$. Furthermore,*

$$\mathcal{O}[X] \cong \bigoplus_{\lambda \in \mathrm{Spec}(X)} E_\lambda \otimes \mathcal{O}[X]^{N^+}(\lambda) \quad \textit{(algebraic direct sum)} \tag{2.3}$$

as a G-module, with action $\pi_\lambda(g) \otimes 1$ on the λ summand.

The action of G on $\mathcal{O}[X]$ is not only linear; it also preserves the algebra structure. Since $\mathcal{O}[X]^{N^+}(\lambda) \cdot \mathcal{O}[X]^{N^+}(\mu) \subset \mathcal{O}[X]^{N^+}(\lambda+\mu)$ under pointwise multiplication and $\mathcal{O}[X]$ has no zero divisors (X is irreducible), it follows from (2.3) that

$$\mathrm{Spec}(X) \text{ is an additive subsemigroup of } P_{++}.$$

The multiplicity of π_λ in $\mathcal{O}[X]$ is $\dim \mathcal{O}[X]^{N^+}(\lambda)$ (which may be infinite). All of this was certainly known (perhaps in less precise form) by Cartan and Weyl at the time [Pe-We] appeared. We now consider Cartan's goal in [Car2] to determine the decomposition (2.3) when G acts transitively on X; especially, when X is a symmetric space. This requires determining the *spectrum* and the *multiplicities* in this decomposition.

6 Weyl uses a similar construction in [Wey2], defining intertwining maps by integration over a compact homogeneous space.

2.2 Multiplicity Free Spaces

We say that an irreducible affine G-space X is *multiplicity free* if all the irreducible representations of G that occur in $\mathcal{O}[X]$ have multiplicity one. Thanks to the theorem of the highest weight, this property can be translated into a geometric statement (see [Vi-Ki]). For a subgroup $K \subset G$ and $x \in X$ write $K_x = \{k \in L : k \cdot x = x\}$ for the isotropy group at x.

Theorem 2.2 (Vinberg–Kimelfeld) *Suppose there is a point $x_0 \in X$ such that $B \cdot x_0$ is open in X. Then X is multiplicity free. In this case, if $\lambda \in \mathrm{Spec}(X)$ then $h^\lambda = 1$ for all $h \in H_{x_0}$.*

Proof If $B \cdot x_0$ is open in X, then it is Zariski dense in X (since X is irreducible). Hence $f \in \mathcal{O}[X]^{N^+}(\lambda)$ is determined by $f(x_0)$, since on the dense set $B \cdot x_0$ it satisfies $f(b \cdot x_0) = b^{-\lambda} f(x_0)$. In particular, if $f \neq 0$ then $f(x_0) \neq 0$, and hence $h^\lambda = 1$ for all $h \in H_{x_0}$. Thus

$$\dim \mathcal{O}[X]^{N^+}(\lambda) \leq 1 \quad \text{for all } \lambda \in P_{++}.$$

Now apply Theorem 2.1. □

Remark. The converse to Theorem 2.2 is true; this depends on some results of Rosenlicht [Ros] and is the starting point for the classification of multiplicity free spaces (see [Be-Ra]).

Example: Algebraic Peter–Weyl Decomposition

Theorem 2.2 implies the algebraic version of the Peter-Weyl decomposition of the regular representation of G. Consider the reductive group $G \times G$ acting on $X = G$ by left and right translations. Denote this representation by ρ:

$$\rho(y, z) f(x) = f(y^{-1} x z), \quad \text{for} \quad f \in \mathcal{O}[G] \ \text{ and } \ x, y, z \in G.$$

Take $H \times H$ as the Cartan subgroup and $\bar{B} \times B$ as the Borel subgroup of $G \times G$. Let $x_0 = I$ (the identity in G). The orbit of x_0 under the Borel subgroup is

$$(\bar{B} \times B) \cdot x_0 = N^- H N^+ \quad \text{(Gauss decomposition)} \tag{2.4}$$

This orbit is open in G since $\mathfrak{g} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$. Hence G is multiplicity free as a $G \times G$ space. The $G \times G$ highest weights (relative to this choice of Borel subgroup) are pairs $(w_0\mu, \lambda)$, with $\lambda, \mu \in P_{++}$. The diagonal subgroup $\widetilde{H} = \{(h, h) \,:\, h \in H\}$ fixes x_0, so if $(w_0\mu, \lambda)$ occurs as a highest weight in $\mathcal{O}[X]$, then

$$h^{w_0\mu+\lambda} = 1 \quad \text{for all } h \in H.$$

This means that $\mu = -w_0\lambda = \lambda^*$; hence $E_\mu = E_{\lambda^*}$ is the contragredient representation of G.

Now set $\psi_\lambda(g) = \langle \pi_\lambda(g)e_\lambda,\, f_{\lambda^*}\rangle$. This function satisfies $\psi_\lambda(x_0) = 1$ and

$$\psi_\lambda(\bar{b}^{-1}gb) = \langle \pi_\lambda(g)\pi_\lambda(b)e_\lambda,\, \pi_{\lambda^*}(\bar{b})f_{\lambda^*}\rangle = b^\lambda \bar{b}^{w_0\lambda^*}\psi_\lambda(g)$$

for $b \in B$ and $\bar{b} \in \bar{B}$. Hence ψ_λ is a $B \times \bar{B}$ highest weight vector for $G \times G$ of weight $(w_0\lambda^*, \lambda)$. This proves that $\operatorname{Spec}(X) = \{(w_0\lambda^*, \lambda) : \lambda \in P_{++}\}$.

Theorem 2.3 *For $\lambda \in P_{++}$ let $V_\lambda = \operatorname{Span}\{\rho(G \times G)\psi_\lambda\}$. Then $V_\lambda \cong E_{\lambda^*} \otimes E_\lambda$ as a $G \times G$ module. Furthermore,*

$$\mathcal{O}[G] = \bigoplus_{\lambda \in P_{++}} V_\lambda. \tag{2.5}$$

In particular, $\mathcal{O}[G]$ is multiplicity free as a $G \times G$ module, while under the action of $G \times 1$ it decomposes into the sum of $\dim E_\lambda$ copies of E_λ for all $\lambda \in P_{++}$.

The function ψ_λ in Theorem 2.3 is called the *generating function* [Žel] for the representation π_λ. Since $\psi_\lambda(n^- h n^+) = h^\lambda$ and $N^- H N^+$ is dense in G, it is clear that

$$\psi_\lambda(g)\psi_\mu(g) = \psi_{\lambda+\mu}(g). \tag{2.6}$$

The semigroup P_{++} of dominant integral weights is free with generators $\lambda_1, \ldots, \lambda_l$, called the *fundamental weights*.

Proposition 2.4 (Product Formula) *Set $\psi_i(g) = \psi_{\lambda_i}(g)$. Let $\lambda \in P_{++}$ and write $\lambda = m_1\lambda_1 + \cdots + m_l\lambda_l$ with $m_i \in \mathbb{N}$. Then*

$$\psi_\lambda(g) = \psi_1(g)^{m_1} \cdots \psi_l(g)^{m_l} \quad \text{for } g \in G. \tag{2.7}$$

Remark. From the product formula it is evident that the existence of a rational representation with highest weight λ is equivalent to the property that the functions $n^- h n^+ \mapsto h^{\lambda_i}$ on $N^- H N^+$ extend to regular functions on G for $i = 1, \ldots, l$.

Example. Suppose $G = \mathrm{SL}(n, \mathbb{C})$. Take B as the group of upper-triangular matrices. We may identify P with $\mathbb{Z}^n$, where $\lambda = [\lambda_1, \ldots, \lambda_n]$ gives the character

$$h^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}, \quad h = \mathrm{diag}[x_1, \ldots, x_n].$$

Then P_{++} consists of the monotone decreasing n-tuples and is generated by

$$\lambda_i = [\underbrace{1, \ldots, 1}_{i}, 0, \ldots, 0] \quad \text{for } i = 1, \ldots, n-1.$$

The fundamental representations are the exterior powers $E_{\lambda_i} = \bigwedge^i \mathbb{C}^n$ of the defining representation, for $i = 1, \ldots, n-1$. The generating function $\psi_i(g)$ is the ith principal minor of g. The Gauss decomposition (2.4) is the familiar LDU matrix factorization from linear algebra, and

$$N^- H N^+ = \{ g \in \mathrm{SL}(n, \mathbb{C}) \, : \, \psi_i(g) \neq 0 \quad \text{for } i = 1, \ldots, n-1 \, \}.$$

Let $K \subset G$ be a subgroup and let $\mathcal{O}[G]^{R(K)}$ be the *right* K-invariant regular functions on G (those functions f such that $f(gk) = f(g)$ for all $k \in K$). This subspace of $\mathcal{O}[G]$ is invariant under left translations by G.

Corollary 2.1 *Let E_λ^K be the subspace of K-fixed vectors in E_λ. Then*

$$\mathcal{O}[G]^{R(K)} \cong \bigoplus_{\lambda \in P_{++}} E_\lambda \otimes E_{\lambda^*}^K \tag{2.8}$$

as a G module under left translations, with G acting by $\pi_\lambda \otimes 1$ on the λ-isotypic summand. Thus the multiplicity of π_λ in $\mathcal{O}[G]^{R(K)}$ is $\dim E_{\lambda^*}^K$.

For any closed subgroup K of G whose Lie algebra is a complex subspace of $\mathfrak{g}$, the coset space G/K is a complex manifold on which G acts holomorphically, and the elements of $\mathcal{O}[G]^{R(K)}$ are holomorphic functions on G/K. When K is a *reductive* algebraic subgroup, then the manifold G/K also has the structure of an affine algebraic G-space such that the regular functions are exactly the elements of $\mathcal{O}[G]^{R(K)}$ (a result of Matsushima [Mat]; see also Borel and Harish-Chandra [Bo-Ha]). Also, when K is reductive then $\dim E_{\lambda^*}^K = \dim E_\lambda^K$, since the identity representation is self-dual.

The pair (G, K) is called *spherical* if

$$\dim E_\lambda^K \leq 1 \quad \text{for all } \lambda \in P_{++}.$$

In this case, we refer to K as a *spherical subgroup* of G. When K is reductive, this property is equivalent to G/K being a multiplicity-free G-space, by Corollary 2.1.

3 Complexifications of Compact Symmetric Spaces

3.1 Algebraic Version of Cartan Embedding

Cartan's paper [Car2] studies the decomposition of $C(U/K_0)$, where U is a compact real form of the simply-connected complex semisimple group G and $K_0 = U^\theta$ is the fixed-point set of an involutive automorphism θ of U. The compact symmetric space $X = U/K_0$ is simply-connected and hence the group K_0 is connected.[7] The involution extends uniquely to an algebraic group automorphism of G that we continue to denote as θ. The algebraic subgroup group $K = G^\theta$ is connected and is the complexification of K_0 in G, hence reductive. By Matsushima's theorem G/K is an affine algebraic variety. It can be embedded into G as an affine algebraic subset as follows (see [Ric1], [Ric2]):

Define

$$g \star y = gy\theta(g)^{-1}, \quad \text{for } g, y \in G.$$

We have $(g \star (h \star y)) = (gh) \star y$ for $g, h, y \in G$, so this gives an action of G on itself which we will call the *θ-twisted conjugation* action. Let

$$Q = \{y \in G : \theta(y) = y^{-1}\}.$$

Then Q is an algebraic subset of G. Since $\theta(g \star y) = \theta(g)y^{-1}g^{-1} = (g \star y)^{-1}$, we have $G \star Q = Q$.

Theorem 3.1 (Richardson) *The θ-twisted action of G is transitive on each irreducible component of Q. Hence Q is a finite union of Zariski-closed θ-twisted G-orbits.*

The proof consists of showing that the tangent space to a twisted G-orbit coincides with the tangent space to Q.

Corollary 3.1 *Let $P = G \star 1 = \{g\theta(g)^{-1} : g \in G\}$ be the orbit of the identity element under the θ-twisted conjugation action. Then P is a Zariski-closed irreducible subset of G isomorphic to G/K as an affine G-space (relative to the θ-twisted conjugation action of G).*

7 This theorem of Cartan extends Weyl's results for compact semisimple groups–see Borel [Bor, Chap. IV, §2].