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Gregory Falkovich. Introduction to turbulence theory.

The emphasis of this short course is on fundamental properties of developed turbulence, weak and strong. We shall be focused on the degree of universality and symmetries of the turbulent state. We shall see, in particular, which symmetries remain broken even when the symmetry-breaking factor goes to zero, and which symmetries, on the contrary, emerge in the state of developed turbulence.

1.1 Introduction

Turba is Latin for crowd and “turbulence” initially meant the disordered movements of large groups of people. Leonardo da Vinci was probably the first to apply the term to the random motion of fluids. In 20th century, the notion has been generalized to embrace far-from-equilibrium states in solids and plasma. We now define turbulence as a state of a physical system with many interacting degrees of freedom deviated far from equilibrium. This state is irregular both in time and in space and is accompanied by dissipation.

We consider here developed turbulence when the scale of the externally excited motions deviate substantially from the scales of the effectively dissipated ones. When fluid motion is excited on the scale L with the typical velocity V, developed turbulence takes place when the Reynolds number is large: $Re = VL/\nu \gg 1$. Here ν is the kinematic viscosity. At large Re, flow perturbations produced at the scale L have their viscous dissipation small compared to the nonlinear effects. Nonlinearity produces motions of smaller and smaller scales until viscous dissipation stops this at a scale much smaller than L so that there is a wide (so-called inertial) interval of scales where viscosity is negligible and nonlinearity dominates. Another example is the system of waves excited on a fluid surface by wind or moving bodies and in plasma and solids by external electromagnetic fields. The state of such system is called wave turbulence when the wavelength of the waves excited strongly differs from the wavelength of the waves that effectively dissipate. Nonlinear interaction excites waves in the interval of wavelengths (called transparency window or inertial interval) between the injection and dissipation scales.

Simultaneous existence of many modes calls for a statistical description based upon averaging either over regions of space or intervals of time. Here we focus on a single-time statistics of steady turbulence that is on the spatial structure of fluctuations in the inertial range. The basic question is that of universality: to what extent the statistics of such fluctuations is independent of the details of external forcing and internal friction and which features are common to different turbulent systems. This quest for universality is motivated by the hope of being able to distinguish general principles that govern far-from-equilibrium systems, similar in scope to the variational principles that govern thermal equilibrium.

Since we generally cannot solve the nonlinear equations that describe turbulence, we try to infer the general properties of turbulence statistics from symmetries or conservation laws. The conservation laws are broken by pumping and dissipation, but both factors do not act directly in the inertial interval. For example, in the incompressible turbulence, the kinetic energy is pumped by a (large-scale) external forcing and is dissipated by viscosity (at small scales). One may suggest that the kinetic energy is transferred from large to small scales in a cascade-like process i.e. the energy flows throughout the inertial interval of scales. The cascade idea (suggested by Richardson in 1921) explains the basic macroscopic manifestation of turbulence: the rate of dissipation of the dynamical integral of motion has a finite limit when the dissipation coefficient tends to zero. For example, the mean rate of the viscous energy dissipation does not depend on viscosity at large Reynolds numbers. Intuitively, one can imagine turbulence cascade as a pipe in wavenumber space that carries energy. As viscosity gets smaller the pipe gets longer but the flux it carries does not change. Formally, that means that the symmetry of the inviscid equation (here, time-reversal invariance) is broken by the presence of the viscous term, even though the latter might have been expected to become negligible in the limit $Re \to \infty$.

One can use the cascade idea to guess the scaling properties of turbulence. For incompressible fluid, the energy flux (per unit mass) ϵ through the given scale r can be estimated via the velocity difference δv measured at that scale as the energy $(\delta v)^2$ divided by the time $r/\delta v$. That gives $(\delta v)^3 \sim \epsilon r$. Of course, δv is a fluctuating quantity and we ought to make statements on its moments or probability distribution $\mathcal{P}(\delta v, r)$. Energy flux constancy fixes the third moment, $\langle(\delta v)^3\rangle \sim \epsilon r$. It is a natural wish to have turbulence scale invariant in the inertial interval so that $\mathcal{P}(\delta v, r) = (\delta v)^{-1} f[\delta v/(\epsilon r)^{1/3}]$ is expressed via the dimensionless function f of a single variable. Initially, Kolmogorov made even stronger wish for the function f to be universal (i.e. pumping independent). Nature is under no obligation to grant wishes of even great scientists, particularly when it is in a state of turbulence. After hearing Kolmogorov talk, Landau remarked that the moments different from third are nonlinear functions of the input rate and must be sensitive to the precise statistics of the pumping. As we show below, the cascade idea can indeed be turned into an exact relation for the simultaneous correlation function which expresses the flux (third or fourth-order moment depending on the degree of nonlinearity). The relation requires the mean flux of the respective integral of motion to be constant across the inertial interval of scales. We shall see that flux constancy determines the system completely only for a weakly nonlinear system (where the statistics is close to Gaussian i.e. not only scale invariant but also perfectly universal). To describe an entire turbulence statistics of strongly interacting systems, one has to solve problems on a case-by-case basis with most cases still unsolved. Particularly difficult (and interesting) are the cases when not only universality but also scale invariance is broken so that knowledge of the flux does not allow one to predict even the order of magnitude of high moments. We describe the new concept of statistical integrals of motion which allows for the description of system with broken scale invariance. We also describe situations when not only scale invariance is restored but a wider conformal invariance takes place in the inertial interval.

1.2 Weak wave turbulence

It is easiest to start from a weakly nonlinear system. Such is a system of small-amplitude waves. Examples include waves on the water surface, waves in plasma with and without magnetic field, spin waves in magnetics etc. We assume spatial homogeneity and denote a_k the amplitude of the wave with the wavevector $\mathbf{k}$. Considering for a moment wave system as closed (that is without external pumping and dissipation) one can describe it as a Hamiltonian system using wave amplitudes as normal canonical variables — see, for instance, the monograph Zakharov et al 1992. At small amplitudes, the Hamiltonian can be written as an expansion over a_k, where the second-order term describes non-interacting waves and high-order terms determine the interaction†:

$$H = \int \omega_k |a_k|^2 \, d\mathbf{k} \tag{1.1}$$
$$+ \int \Big(V_{123} a_1 a_2^* a_3^* + c.c. \Big) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \, d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 + \mathrm{O}(a^4)\,.$$

The dispersion law ω_k describes wave propagation, $V_{123} = V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ is the interaction vertex and c.c. means complex conjugation. In the Hamiltonian expansion, we presume every subsequent term smaller than the previous one, in particular, $\xi_k = |V_{kkk} a_k| k^d / \omega_k \ll 1$ — wave turbulence that satisfies that condition is called weak turbulence. Here d is the space dimensionality.

The dynamic equation which accounts for pumping, damping, wave propagation and interaction has the following form:

$$\partial a_k / \partial t = -i \delta H / \delta a_k^* + f_k(t) - \gamma_k a_k \ . \tag{1.2}$$

Here γ_k is the decrement of linear damping and f_k describes pumping. For a linear system, a_k is different from zero only in the regions of $\mathbf{k}$-space where f_k is nonzero. Nonlinear interaction provides for wave excitation outside pumping regions.

It is likely that the statistics of the weak turbulence at $k \gg k_f$ is close to Gaussian for wide classes of pumping statistics. When the forcing $f_k(t)$ is Gaussian then the statistics of $a_k(t)$ is close to Gaussian as long as nonlinearity is weak. However, in most cases in nature and in the lab, the force is not Gaussian even though its amplitude can be small. It is an open problem to find out under what conditions the wave field is close to Gaussian with $\langle a_k(0) a_{k'}^*(t) \rangle = n_k \exp(-\imath \omega_k t) \delta(\mathbf{k} + \mathbf{k}')$. This problem actually breaks into two parts. The first one is to solve the linear equation for the waves in the spectral interval of pumping and formulate the criteria on the forcing that guarantee that the cumulants are small for $a_k(t) = \exp(-\imath \omega_k t - \gamma_k t) \int_0^t f_k(t') \exp(\imath \omega_k t + \gamma_k t) \, dt'$. The second part is more interesting: even when the pumping-related waves are non-Gaussian, it may well be that as we go in k-space away from pumping (into the inertial interval) the field $a_k(t)$ is getting more Gaussian. Unless we indeed show that, most of the applications of the weak turbulence theory described in this section are in doubt. See also Choi et al 2005.

We consider here and below a pumping by a Gaussian random force statistically isotropic and homogeneous in space, and white in time (see also

† For example, for sound one expands the (kinetic plus internal) energy density $\rho v^2/2 + E(\rho)$ assuming $v \ll c$ and using $\mathbf{v}_k = \mathbf{k}(a_k - a_{-k}^*)(ck/2\rho_0)^{1/2}$, $\rho_k = k(a_k + a_{-k}^*)(\rho_0/2ck)^{1/2}$.

Sect. 3.1 of John Cardy's course):

$$\langle f_k(t) f^*_{k'}(t')\rangle = F(k)\delta(\mathbf{k}+\mathbf{k}')\delta(t-t') \ . \tag{1.3}$$

Angular brackets mean spatial average. We assume $\gamma_k \ll \omega_k$ (for waves to be well defined) and that $F(k)$ is nonzero only around some k_f.

As long as we assume the statistics of the wave system to be close to Gaussian, it is completely determined by the pair correlation function. Here we are interesting in the spatial structure which is described by the single-time pair correlation function $\langle a_k(t) a^*_{k'}(t)\rangle = n_k(t)\delta(\mathbf{k}+\mathbf{k}')$. Since the dynamic equation (1.2) contains a quadratic nonlinearity then the time derivative of the second moment, $\partial n_k/\partial t$, is expressed via the third one, the time derivative of the third moment ix expressed via the fourth one etc; that is the statistical description in terms of moments encounters the closure problem. Fortunately, weak turbulence in the inertial interval is expected to have the statistics close to Gaussian so one can express the fourth moment as the product of two second ones. As a result one gets a closed equation (see e.g. Zakharov *et al* 1992):

$$\frac{\partial n_k}{\partial t} = F_k - \gamma_k n_k + I_k^{(3)} , \qquad I_k^{(3)} = \int \big(U_{k12} - U_{1k2} - U_{2k1}\big)\, d\mathbf{k}_1 d\mathbf{k}_2 \tag{1.4}$$
$$U_{123} = \pi\big[n_2 n_3 - n_1(n_2+n_3)\big]|V_{123}|^2 \delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega_1-\omega_2-\omega_3) \ .$$

It is called kinetic equation for waves. The collision integral $I_k^{(3)}$ results from the cubic terms in the Hamiltonian i.e. from the quadratic terms in the equations for amplitudes. It can be *interpreted* as describing three-wave interactions: the first term in the integral (1.4) corresponds to a decay of a given wave while the second and third ones to a confluence with other wave.

The inverse time of nonlinear interaction at a given k can be estimated from (1.4) as $|V(k,k,k)|^2 n(k) k^d/\omega(k)$. We define the dissipation wavenumber k_d as such where this inverse time is comparable to $\gamma(k_d)$ and assume nonlinearity to dominate over dissipation at $k \ll k_d$. As has been noted, wave turbulence appears when there is a wide (inertial) interval of scales where both pumping and damping are negligible, which requires $k_d \gg k_f$, the condition analogous to $Re \gg 1$. This is schematically shown in Fig. 1.

The presence of frequency delta-function in $I_k^{(3)}$ means that in the first order of perturbation theory in wave interaction we account only for resonant processes which conserve the quadratic part of the energy $E = \int \omega_k n_k \, d\mathbf{k} = \int E_k dk$. For the cascade picture to be valid, the collision integral has to converge in the inertial interval which means that energy exchange is small between motions of vastly different scales, the property called interaction locality in k-space (see the exercise 1.1 below). Consider now a statistical

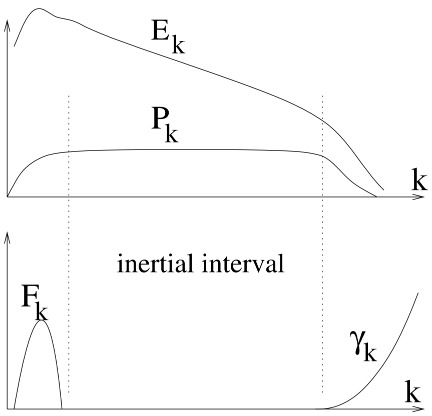

Fig. 1.1. A schematic picture of the cascade.

steady state established under the action of pumping and dissipation. Let us multiply (1.4) by ω_k and integrate it over either interior or exterior of the ball with radius k. Taking $k_f \ll k \ll k_d$, one sees that the energy flux through any spherical surface (Ω is a solid angle), is constant in the inertial interval and is equal to the energy production/dissipation rate ϵ:

$$P_k = \int_0^k k^{d-1} dk \int d\Omega \, \omega_k I_k^{(3)} = \int \omega_k F_k \, d\mathbf{k} = \int \gamma_k E_k \, dk = \epsilon \ . \qquad (1.5)$$

That (integral) equation determines n_k. Let us assume now that the medium (characterized by the Hamiltonian coefficients) can be considered isotropic at the scales in the inertial interval. In addition, for scales much larger or much smaller than a typical scale (like Debye radius in plasma or the depth of the water) the medium is usually scale invariant: $\omega(k) = ck^\alpha$ and $|V(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)|^2 = V_0^2 k^{2m} \chi(\mathbf{k}_1/k, \mathbf{k}_2/k)$ with $\chi \simeq 1$. Remind that we presumed statistically isotropic force. In this case, the pair correlation function that describes a steady cascade is also isotropic and scale invariant:

$$n_k \simeq \epsilon^{1/2} V_0^{-1} k^{-m-d} \ . \qquad (1.6)$$

One can show that (1.6), called Zakharov spectrum, turns $I_k^{(3)}$ into zero (see the exercise 1.1 below and Zakharov *et al* 1992).

If the dispersion relation $\omega(k)$ does not allow for the resonance condition $\omega(k_1) + \omega(k_2) = \omega(|\mathbf{k}_1 + \mathbf{k}_2|)$ then the three-wave collision integral is zero and one has to account for four-wave scattering which is always resonant, that is whatever $\omega(k)$ one can always find four wavevectors that satisfy $\omega(k_1)+\omega(k_2)=\omega(k_3)+\omega(k_4)$ and $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4$. The collision integral that describes scattering,

$$I_k^{(4)} = \frac{\pi}{2}\int |T_{k123}|^2\left[n_2n_3(n_1+n_k)-n_1n_k(n_2+n_3)\right]\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \times\delta(\omega_k+\omega_1-\omega_2-\omega_2)\,d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3\ , \tag{1.7}$$

conserves the energy and the wave action $N=\int n_k\,d\mathbf{k}$ (the number of waves). Pumping generally provides for an input of both E and N. If there are two inertial intervals (at $k\gg k_f$ and $k\ll k_f$), then there should be two cascades. Indeed, if $\omega(k)$ grows with k then absorbing finite amount of E at $k_d\to\infty$ corresponds to an absorption of an infinitely small N. It is thus clear that the flux of N has to go in opposite direction that is to large scales. A so-called inverse cascade with the constant flux of N can thus be realized at $k\ll k_f$. A sink at small k can be provided by wall friction in the container or by long waves leaving the turbulent region in open spaces (like in sea storms). Two-cascade picture can be illustrated by a simple example with a wave source at $\omega=\omega_2$ generating N_2 waves per unit time and two sinks at $\omega=\omega_1$ and $\omega=\omega_3$ absorbing respectively N_1 and N_3. In a steady state, $N_2=N_1+N_3$ and $\omega_2N_2=\omega_1N_1+\omega_3N_3$, which gives

$$N_1=N_2\frac{\omega_3-\omega_2}{\omega_3-\omega_1},\qquad N_3=N_2\frac{\omega_2-\omega_1}{\omega_3-\omega_1}\ .$$

At a sufficiently large left inertial interval (when $\omega_1\ll\omega_2<\omega_3$), the whole energy is absorbed by the right sink: $\omega_2N_2\approx\omega_3N_3$. Similarly, at $\omega_3\gg\omega_2>\omega_1$, we have $N_1\approx N_2$, i.e. the wave action is absorbed at small ω.

The collision integral $I_k^{(3)}$ involved products of two n_k so that flux constancy required $E_k\propto\epsilon^{1/2}$ while for the four-wave case $I_k^{(4)}\propto n^3$ gives $E_k\propto\epsilon^{1/3}$. In many cases (when there is a complete self-similarity) that knowledge is sufficient to obtain the scaling of E_k from a dimensional reasoning without actually calculating V and T. For example, short waves on a deep water are characterized by the surface tension σ and density ρ so the dispersion relation must be $\omega_k\sim\sqrt{\sigma k^3/\rho}$ which allows for the three-wave resonance and thus $E_k\sim\epsilon^{1/2}(\rho\sigma)^{1/4}k^{-7/4}$. For long waves on a deep water, the surface-restoring force is dominated by gravity so that the gravity acceleration g replaces σ as a defining parameter and $\omega_k\sim\sqrt{gk}$. Such dispersion law does not allow for the three-wave resonance so that the dominant interaction is four-wave scattering which permits two cascades. The direct energy cascade corresponds to $E_k\sim\epsilon^{1/3}\rho^{2/3}g^{1/2}k^{-5/2}$. The inverse cascade carries the flux of N which we denote Q, it has the dimensionality $[Q] = [\epsilon]/[\omega_k]$ and corresponds to $E_k \sim Q^{1/3}\rho^{2/3}g^{2/3}k^{-7/3}$.

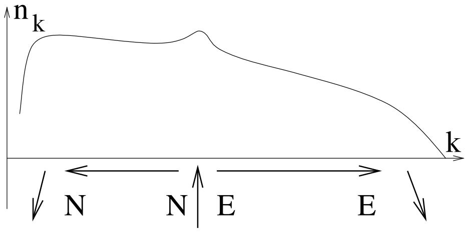

Fig. 1.2. Two cascades under four-wave interaction.

Under a weakly anisotropic pumping, stationary spectrum acquires a small stationary weakly anisotropic correction $\delta n(\mathbf{k})$ such that $\delta n(\mathbf{k})/\mathbf{n_0}(\mathbf{k}) \propto \omega(\mathbf{k})/\mathbf{k}$ (see exercise 2.2). The degree of anisotropy increases with k for waves with the decay dispersion law. That is the spectrum of the weak turbulence generated by weakly anisotropic pumping is getting more anisotropic as we go into the inertial interval of scales. We see that the conservation of the second integral (momentum) can lead to the non-restoration of symmetry (isotropy) in the inertial interval.

Since the statistics of weak turbulence is near Gaussian, it is completely determined by the pair correlation function, which is in turn determined by the respective flux (or fluxes). We thus conclude that weak turbulence is perfectly universal: deep in the inertial interval it "forgets" all the properties of pumping except the flux value.

1.3 Strong wave turbulence

Weak turbulence theory breaks down when the wave amplitudes are large enough (so that $\xi_k \geq 1$). We need special consideration also in the particular case of the linear (acoustic) dispersion relation $\omega(k) = ck$ for arbitrarily small amplitudes (as long as the Reynolds number remains large). Indeed, there is no dispersion of wave velocity for acoustic waves so that waves moving at the same direction interact strongly and produce shock waves when viscosity is small. Formally, there is a singularity due to coinciding arguments of delta-functions in (1.4) (and in the higher terms of perturbation expansion for $\partial n_k/\partial t$), which is thus invalid at however small amplitudes.

Still, some features of the statistics of acoustic turbulence can be understood even without a closed description. We discuss that in a one-dimensional case which pertains, for instance, to sound propagating in long pipes. Since weak shocks are stable with respect to transversal perturbations (Landau and Lifshits 1987), quasi one-dimensional perturbations may propagate in 2d and 3d as well. In the reference moving with the sound velocity, the weakly compressible 1d flows ($u \ll c$) are described by the Burgers equation (Landau and Lifshits 1987, E et al 1997, Frisch and Bec 2001):

$$u_t + uu_x - \nu u_{xx} = 0 \ . \tag{1.8}$$

Burgers equation has a propagating shock-wave solution $u = 2v\{1 + \exp[v(x - vt)/\nu]\}^{-1}$ with the energy dissipation rate $\nu \int u_x^2\, dx$ independent of ν. The shock width ν/v is a dissipative scale and we consider acoustic turbulence produced by a pumping correlated on much larger scales (for example, pumping a pipe from one end by frequencies much less than cv/ν). After some time, it will develop shocks at random positions. Here we consider the single-time statistics of the Galilean invariant velocity difference $\delta u(x,t) = u(x,t) - u(0,t)$. The moments of δu are called structure functions $S_n(x,t) = \langle [u(x,t) - u(0,t)]^n \rangle$. Quadratic nonlinearity relates the time derivative of the second moment to the third one:

$$\frac{\partial S_2}{\partial t} = -\frac{\partial S_3}{3\partial x} - 4\epsilon + \nu \frac{\partial^2 S_2}{\partial x^2} \ . \tag{1.9}$$

Here $\epsilon = \nu \langle u_x^2 \rangle$ is the mean energy dissipation rate. Equation (1.9) describes both a free decay (then ϵ depends on t) and the case of a permanently acting pumping which generates turbulence statistically steady at scales less than the pumping length. In the first case, $\partial S_2/\partial t \simeq S_2 u/L \ll \epsilon \simeq u^3/L$ (where L is a typical distance between shocks) while in the second case $\partial S_2/\partial t = 0$ so that $S_3 = 12\epsilon x + \nu \partial S_2/\partial x$.

Consider now limit $\nu \to 0$ at fixed x (and t for decaying turbulence). Shock dissipation provides for a finite limit of ϵ at $\nu \to 0$ then

$$S_3 = -12\epsilon x \ . \tag{1.10}$$

This formula is a direct analog of (1.5). Indeed, the Fourier transform of (1.9) describes the energy density $E_k = \langle |u_k|^2 \rangle / 2$ which satisfies the equation $(\partial_t - \nu k^2) E_k = -\partial P_k / \partial k$ where the k-space flux

$$P_k = \int_0^k dk' \int_{-\infty}^{\infty} dx S_3(x) k' \sin(k'x)/24 \ .$$

It is thus the flux constancy that fixes $S_3(x)$ which is universal that is determined solely by ϵ and depends neither on the initial statistics for decay nor on the pumping for steady turbulence. On the contrary, other structure functions $S_n(x)$ are not given by $(\epsilon x)^{n/3}$. Indeed, the scaling of the structure functions can be readily understood for any dilute set of shocks (that is when shocks do not cluster in space) which seems to be the case both for smooth initial conditions and large-scale pumping in Burgers turbulence. In this case, $S_n(x) \sim C_n|x|^n + C'_n|x|$ where the first term comes from the regular (smooth) parts of the velocity (the right x-interval in Fig. 1.3) while the second comes from $O(x)$ probability to have a shock in the interval x. The scaling exponents, $\xi_n = d\ln S_n/d\ln x$, thus behave as follows: $\xi_n = n$ for $n \le 1$ and $\xi_n = 1$ for $n > 1$. That means that the probability density

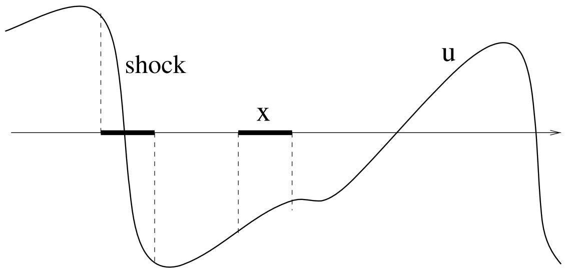

Fig. 1.3. Typical velocity profile in Burgers turbulence.

function (PDF) of the velocity difference in the inertial interval $P(\delta u, x)$ is not scale-invariant, that is the function of the re-scaled velocity difference $\delta u/x^a$ cannot be made scale-independent for any a. Simple bi-modal nature of Burgers turbulence (shocks and smooth parts) means that the PDF is actually determined by two (non-universal) functions, each depending of a single argument: $P(\delta u, x) = \delta u^{-1} f_1(\delta u/x) + x f_2(\delta u/u_{rms})$. Breakdown of scale invariance means that the low-order moments decrease faster than the high-order ones as one goes to smaller scales, i.e. the smaller the scale the more probable are large fluctuations. In other words, the level of fluctuations increases with the resolution. When the scaling exponents ξ_n do not lie on a straight line, this is called an anomalous scaling since it is related again to the symmetry (scale invariance) of the PDF broken by pumping and not restored even when $x/L \to 0$.

As an alternative to the description in terms of structures (shocks), one can relate the anomalous scaling in Burgers turbulence to the additional integrals of motion. Indeed, the integrals $E_n = \int u^{2n}\, dx/2$ are all conserved by the inviscid Burgers equation. Any shock dissipates the finite amount of