Cambridge University Press
978-0-521-71424-2 - A First Course in Continuum Mechanics
Oscar Gonzalez and Andrew M. Stuart
Excerpt
More information

1

Tensor Algebra

Underlying everything we study is the field of real numbers $\mathbb{R}$ and three-dimensional Euclidean space $\mathbb{E}^3$. We refer to elements of $\mathbb{R}$ as **scalars** and denote them by light-face symbols such as a, b, x and y. We refer to elements of $\mathbb{E}^3$ as **points** and denote them by bold-face symbols such as $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{x}$ and $\boldsymbol{y}$.

The aim of this chapter is to build up a mathematical structure for $\mathbb{E}^3$ and various other sets associated with it, such as the set of vectors $\mathcal{V}$. The important ideas that we introduce are: (i) the distinction between points and vectors; (ii) the distinction between a vector and its representation in a coordinate frame; (iii) index notation; (iv) the set of second-order tensors $\mathcal{V}^2$ as linear transformations in $\mathcal{V}$; (v) the set of fourth-order tensors $\mathcal{V}^4$ as linear transformations in $\mathcal{V}^2$.

1.1 Vectors

By a **vector** we mean a quantity with a specified magnitude and direction in three-dimensional space. For example, a line segment in space can be interpreted as a vector as can other things such as forces, velocities and accelerations. We denote a vector by a bold-face symbol such as $\boldsymbol{v}$ and denote its magnitude by $|\boldsymbol{v}|$. Notice that for any vector $\boldsymbol{v}$ we have $|\boldsymbol{v}| \geq 0$. It is useful to define a **zero vector**, $\mathbf{0}$, as a vector with zero magnitude and no specific direction. Any vector having unit magnitude is called a **unit vector**.

A vector may be represented graphically by an arrow as depicted in Figure 1.1. The orientation of the arrow represents the direction of the vector and the length of the arrow represents the magnitude. We will always identify vectors with arrows in this way. Moreover, two vectors

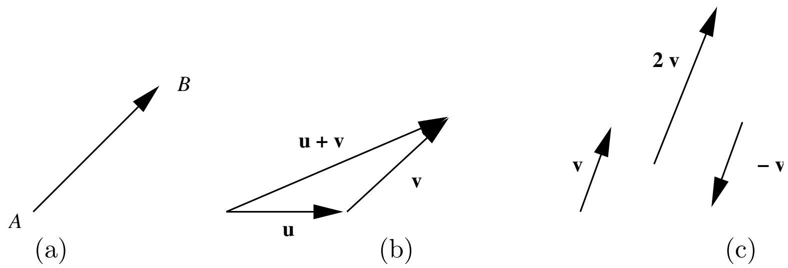

Fig. 1.1 Graphical illustration of a vector and basic vector operations. (a) the tail A and tip B of a vector. (b) the sum of two vectors. (c) the multiplication of a vector by a scalar.

will be considered equal if they have the same direction and magnitude, regardless of their location in space.

Scalars and vectors are examples of more general objects called **tensors**. For example, a scalar is a zeroth-order tensor, while a vector is a first-order tensor. Later we will consider second-order and fourth-order tensors, but for now we concentrate on first-order ones, that is, vectors.

1.1.1 Vector Algebra

If we let $\mathcal{V}$ be the set of all vectors, then $\mathcal{V}$ has the structure of a real vector space since

$$\boldsymbol{u}+\boldsymbol{v}\in\mathcal{V}\quad\forall\boldsymbol{u},\boldsymbol{v}\in\mathcal{V}\quad\text{and}\quad\alpha\boldsymbol{v}\in\mathcal{V}\quad\forall\alpha\in\mathbb{R},\ \boldsymbol{v}\in\mathcal{V}.$$

In particular, we define the sum $\boldsymbol{u}+\boldsymbol{v}$ as that element of $\mathcal{V}$ which completes the triangle when $\boldsymbol{u}$ and $\boldsymbol{v}$ are placed tip-to-tail as shown in Figure 1.1(b). Given any scalar $\alpha\in\mathbb{R}$ we define the product $\alpha\boldsymbol{v}$ as that element in $\mathcal{V}$ with magnitude $|\alpha||\boldsymbol{v}|$, where $|\alpha|$ denotes the absolute value of α, and direction the same as or opposite to $\boldsymbol{v}$ depending on whether α is positive or negative, respectively, as depicted in Figure 1.1(c). If $\alpha=0$, then $\alpha\boldsymbol{v}$ is the zero vector.

With the vector space $\mathcal{V}$ in hand we can define some useful mathematical operations between elements of $\mathbb{E}^3$ and $\mathcal{V}$. Given any two points $\boldsymbol{x}$ and $\boldsymbol{y}$ we define their difference $\boldsymbol{y}-\boldsymbol{x}$ to be the unique vector which points from $\boldsymbol{x}$ to $\boldsymbol{y}$, as shown in Figure 1.2(a). Hence given $\boldsymbol{x},\boldsymbol{y}\in\mathbb{E}^3$ we have $\boldsymbol{y}-\boldsymbol{x}\in\mathcal{V}$. Given any point $\boldsymbol{z}$ and any vector $\boldsymbol{v}$ we define the sum $\boldsymbol{z}+\boldsymbol{v}$ to be the unique point such that $(\boldsymbol{z}+\boldsymbol{v})-\boldsymbol{z}=\boldsymbol{v}$, as shown in Figure 1.2(b). Hence given $\boldsymbol{z}\in\mathbb{E}^3$ and $\boldsymbol{v}\in\mathcal{V}$ we have $\boldsymbol{z}+\boldsymbol{v}\in\mathbb{E}^3$.

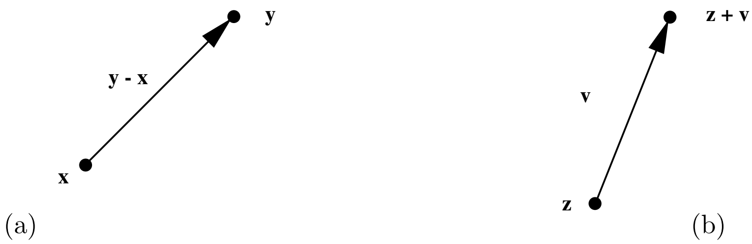

Fig. 1.2 Graphical interpretation of operations between points and vectors. (a) the difference between two points is a vector. (b) the sum of a point and a vector is a point.

We next define some geometrical operations between vectors which will allow us to construct mathematical representations for $\mathcal{V}$ and $\mathbb{E}^3$.

1.1.2 Scalar and Vector Products

The **scalar** or **dot product** of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined geometrically as

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta,$$

where $\theta \in [0, \pi]$ is the angle between the tips of $\boldsymbol{a}$ and $\boldsymbol{b}$. Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be **orthogonal** or **perpendicular** if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. For any vector $\boldsymbol{a}$ notice that $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.

The **vector** or **cross product** of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined geometrically as

$$\boldsymbol{a} \times \boldsymbol{b} = (|\boldsymbol{a}||\boldsymbol{b}| \sin\theta)\boldsymbol{e},$$

where, as before, $\theta \in [0, \pi]$ is the angle between the tips of $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{e}$ is a unit vector perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$. The orientation of $\boldsymbol{e}$ is defined such that a right-handed rotation about $\boldsymbol{e}$, through an angle θ, carries $\boldsymbol{a}$ to $\boldsymbol{b}$. If $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be **parallel**. Notice that the magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is the same as the area of a parallelogram with sides defined by $\boldsymbol{a}$ and $\boldsymbol{b}$.

Example: Let $V = |\boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}|$. Then V is the volume of the parallelepiped defined by $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. In particular, we have

$$V = |\boldsymbol{a}||\boldsymbol{b}||\boldsymbol{c}|| \sin\theta|| \cos\phi|,$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, and ϕ is the angle between $\boldsymbol{c}$ and

$\boldsymbol{a}\times\boldsymbol{b}$. The factor $|\boldsymbol{a}||\boldsymbol{b}||\sin\theta|$ is the area of the parallelogram defined by $\boldsymbol{a}$ and $\boldsymbol{b}$, and the factor $|\boldsymbol{c}||\cos\phi|$ gives the height of the parallelepiped with respect to the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$. □

1.1.3 Projections, Bases and Coordinate Frames

Let $\boldsymbol{e}$ be a unit vector. Then any vector $\boldsymbol{v}$ can be written in the form

$$\boldsymbol{v} = \boldsymbol{v}_e + \boldsymbol{v}_e^\perp, \tag{1.1}$$

where $\boldsymbol{v}_e$ is parallel to $\boldsymbol{e}$ and $\boldsymbol{v}_e^\perp$ is perpendicular to $\boldsymbol{e}$. The vector $\boldsymbol{v}_e$ is called the **projection** of $\boldsymbol{v}$ parallel to $\boldsymbol{e}$ and is defined as

$$\boldsymbol{v}_e = (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}.$$

The vector $\boldsymbol{v}_e^\perp$ is called the projection of $\boldsymbol{v}$ perpendicular to $\boldsymbol{e}$. Since $\boldsymbol{v}_e^\perp = \boldsymbol{v} - \boldsymbol{v}_e$ we find that

$$\boldsymbol{v}_e^\perp = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}.$$

Thus any vector $\boldsymbol{v}$ can be uniquely decomposed into parts parallel and perpendicular to a given unit vector $\boldsymbol{e}$.

By a right-handed orthonormal **basis** for $\mathcal{V}$ we mean three mutually perpendicular unit vectors $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ which are oriented in the sense that

$$\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}, \qquad \boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}, \qquad \boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}.$$

While any three mutually perpendicular unit vectors necessarily satisfy $|(\boldsymbol{i}\times\boldsymbol{j})\cdot\boldsymbol{k}| = 1$, a right-handed basis has the property that $(\boldsymbol{i}\times\boldsymbol{j})\cdot\boldsymbol{k} = 1$. In contrast, a left-handed basis satisfies $(\boldsymbol{i}\times\boldsymbol{j})\cdot\boldsymbol{k} = -1$.

Using the notation $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ in place of $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ we find, by repeated application of (1.1), that any vector $\boldsymbol{v}$ can be uniquely decomposed as

$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 \quad \text{where} \quad v_i = \boldsymbol{v}\cdot\boldsymbol{e}_i \in \mathbb{R}.$$

The numbers v_1, v_2, v_3 are called the **components** of $\boldsymbol{v}$ in the given basis $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$. For brevity we will often denote the set of components by v_i and a given basis by $\{\boldsymbol{e}_i\}$, where it is understood that the subscript i ranges from 1 to 3.

We will frequently find it useful to arrange the components of a vector $\boldsymbol{v}$ into a 3×1 column matrix $[\boldsymbol{v}]$, in particular

$$[\boldsymbol{v}] = \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \in \mathbb{R}^3,$$

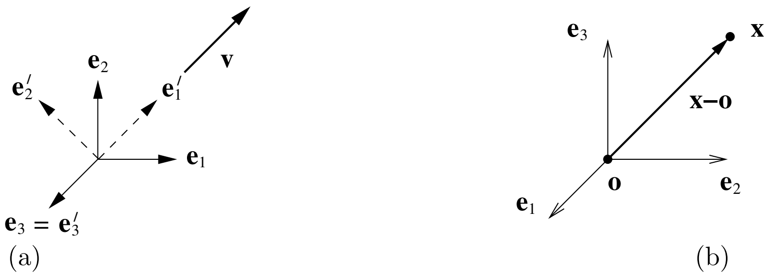

Fig. 1.3 Graphical illustration of orthonormal bases and a coordinate frame. (a) two orthonormal bases for $\mathcal{V}$. (b) Cartesian coordinate frame for $\mathbb{E}^3$.

where $\mathbb{R}^3$ denotes the set of all ordered triplets of real numbers. We will also have need to consider the **transpose** of $[\boldsymbol{v}]$, which is a 1×3 row matrix defined as $[\boldsymbol{v}]^T = (v_1, v_2, v_3)$. Notice that by properties of the transpose we may write $[\boldsymbol{v}] = (v_1, v_2, v_3)^T$. We call $[\boldsymbol{v}]$ the matrix **representation** of $\boldsymbol{v}$ in the given basis.

It is important to distinguish between vectors $\boldsymbol{v} \in \mathcal{V}$ (arrows in space) and their representations $[\boldsymbol{v}] \in \mathbb{R}^3$ (triplets of numbers). In particular, a given vector can have different representations depending on the choice of basis for $\mathcal{V}$.

Example: Let $\{\boldsymbol{e}_i\}$ be a basis for $\mathcal{V}$ and consider a vector $\boldsymbol{v}$ whose representation in this basis is $[\boldsymbol{v}] = (1, 1, 0)^T$ as shown in Figure 1.3(a). Thus

$$\boldsymbol{v} \cdot \boldsymbol{e}_1 = 1, \ \boldsymbol{v} \cdot \boldsymbol{e}_2 = 1, \ \boldsymbol{v} \cdot \boldsymbol{e}_3 = 0 \quad \text{or} \quad \boldsymbol{v} = 1\,\boldsymbol{e}_1 + 1\,\boldsymbol{e}_2 + 0\,\boldsymbol{e}_3.$$

Next, consider the same vector $\boldsymbol{v}$, but rotate the basis $\{\boldsymbol{e}_i\}$ through an angle of $\pi/4$ about the axis defined by $\boldsymbol{e}_3$. The result of this rotation is a new basis $\{\boldsymbol{e}'_i\}$, and in this basis the components of $\boldsymbol{v}$ are

$$\boldsymbol{v} \cdot \boldsymbol{e}'_1 = \sqrt{2}, \ \boldsymbol{v} \cdot \boldsymbol{e}'_2 = 0, \ \boldsymbol{v} \cdot \boldsymbol{e}'_3 = 0 \quad \text{or} \quad \boldsymbol{v} = \sqrt{2}\,\boldsymbol{e}'_1 + 0\,\boldsymbol{e}'_2 + 0\,\boldsymbol{e}'_3.$$

The representation of $\boldsymbol{v}$ in the new basis is $[\boldsymbol{v}]' = (\sqrt{2}, 0, 0)^T$ and we see that $[\boldsymbol{v}]' \neq [\boldsymbol{v}]$. □

By a Cartesian **coordinate frame** for $\mathbb{E}^3$ we mean a reference point $\boldsymbol{o} \in \mathbb{E}^3$ called an **origin** together with a right-handed orthonormal basis $\{\boldsymbol{e}_i\}$ for the associated vector space $\mathcal{V}$ as depicted in Figure 1.3(b). To any point $\boldsymbol{x} \in \mathbb{E}^3$ we then ascribe **coordinates** x_i according to the expression

$$x_i = (\boldsymbol{x} - \boldsymbol{o}) \cdot \boldsymbol{e}_i.$$

Thus the coordinates of $\boldsymbol{x}$ are the unique numbers x_i such that

$$\boldsymbol{x} - \boldsymbol{o} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3.$$

Throughout our developments we will use the term coordinate frame for $\mathbb{E}^3$ without making explicit reference to the point $\boldsymbol{o} \in \mathbb{E}^3$ chosen as the origin. Moreover, we will frequently identify a point $\boldsymbol{x} \in \mathbb{E}^3$ with its position vector $\boldsymbol{x} - \boldsymbol{o} \in \mathcal{V}$ and use the same symbol $\boldsymbol{x}$ for both.

1.2 Index Notation

In this section we express various operations between vectors in terms of their components in a given frame. We begin with the summation convention for index notation and then introduce the Kronecker delta and permutation symbols which will be used throughout the remainder of our developments.

1.2.1 Summation Convention

The representation of vectors and vector operations in terms of components naturally involves sums. For example, let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors with components a_i and b_i in a frame $\{\boldsymbol{e}_i\}$. Then

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 = \sum_{i=1}^{3} a_i\boldsymbol{e}_i,$$

$$\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 = \sum_{j=1}^{3} b_j\boldsymbol{e}_j,$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = \left(\sum_{i=1}^{3} a_i\boldsymbol{e}_i\right)\cdot\left(\sum_{j=1}^{3} b_j\boldsymbol{e}_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \boldsymbol{e}_i\cdot\boldsymbol{e}_j = \sum_{i=1}^{3} a_i b_i,$$

where the last line follows from the fact that $\boldsymbol{e}_i\cdot\boldsymbol{e}_j = 1$ or 0 depending on whether $i = j$ or $i \neq j$.

The expressions for $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}\cdot\boldsymbol{b}$ each involves a sum over a pair of indices. Because sums of this type occur so often we adopt a convention and abbreviate the above three equations as

$$\boldsymbol{a} = a_i\boldsymbol{e}_i, \quad \boldsymbol{b} = b_j\boldsymbol{e}_j \quad \text{and} \quad \boldsymbol{a}\cdot\boldsymbol{b} = a_i b_i.$$

The convention is: *whenever an index occurs twice in a term a sum is implied over that index.* In particular, $a_i\boldsymbol{e}_i$ is shorthand for the sum $\sum_{i=1}^{3} a_i\boldsymbol{e}_i$. Unless mentioned otherwise we will always assume that the summation convention is in force. In this case any pair of repeated indices in a term represents a sum, for example

$$a_2 b_j b_j b_3 = a_2(b_1 b_1 + b_2 b_2 + b_3 b_3)b_3,$$
$$a_i b_i + b_i b_i = (a_1 b_1 + a_2 b_2 + a_3 b_3) + (b_1 b_1 + b_2 b_2 + b_3 b_3),$$
$$a_i a_j b_j a_i = a_i a_i a_j b_j = (a_1 a_1 + a_2 a_2 + a_3 a_3)(a_1 b_1 + a_2 b_2 + a_3 b_3).$$

Any repeated index, such as i or j above, which is used to represent a sum is called a **dummy index**. This terminology reflects the fact that the actual symbol used for a repeated index is immaterial since

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_j b_j.$$

Notice that the summation convention applies only to *pairs* of repeated indices. In particular, no sums will be implied in expressions of the form a_i, $a_i b_i b_i$, and so on.

Any index which is not a dummy index is called a **free index**. For example, in the equation

$$a_i = c_j b_j b_i,$$

the index i is a free index, while j is a dummy index. Free indices can take any of the values 1, 2 or 3, and are used to abbreviate groups of similar equations. For example, the above equation is shorthand for the three equations

$$a_1 = c_j b_j b_1, \qquad a_2 = c_j b_j b_2, \qquad a_3 = c_j b_j b_3.$$

Similarly, the equation

$$a_i b_j = c_i c_k c_k d_j$$

is shorthand for nine equations since both i and j are free indices. The nine equations are

$$a_1 b_1 = c_1 c_k c_k d_1, \qquad a_1 b_2 = c_1 c_k c_k d_2, \qquad \dots \qquad a_3 b_3 = c_3 c_k c_k d_3.$$

Notice that each term in an equation should have the same free indices, and the same symbol cannot be used for both a dummy and free index. For example, equations of the form

$$a_i = b_j, \qquad a_i b_j = c_i d_j d_j \qquad \text{and} \qquad a_i b_j = c_i c_k d_k d_j + d_p c_l c_l d_q$$

are all ambiguous and are not permissible in our use of the summation convention.

1.2.2 Kronecker Delta and Permutation Symbols

To any frame $\{\boldsymbol{e}_i\}$ we associate a **Kronecker delta** symbol δ_{ij} defined by

$$\delta_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and a **permutation symbol** ϵ_{ijk} defined by

$$\epsilon_{ijk} = (\boldsymbol{e}_i \times \boldsymbol{e}_j) \cdot \boldsymbol{e}_k = \begin{cases} 1, & \text{if } ijk = 123,\ 231 \text{ or } 312, \\ -1, & \text{if } ijk = 321,\ 213 \text{ or } 132, \\ 0, & \text{otherwise (repeated index).} \end{cases}$$

Notice that the numerical values of δ_{ij} and ϵ_{ijk} are the same for any right-handed orthonormal frame.

The Kronecker delta and permutation symbols enjoy certain symmetry properties under index transposition and permutation. In particular, we find that δ_{ij} is invariant under transposition of indices in the sense that $\delta_{ij} = \delta_{ji}$, whereas ϵ_{ijk} changes sign under (pairwise) transposition, namely, $\epsilon_{ijk} = -\epsilon_{jik}$, $\epsilon_{ijk} = -\epsilon_{ikj}$ and $\epsilon_{ijk} = -\epsilon_{kji}$. However, ϵ_{ijk} is invariant under circular or cyclic permutation of indices in the sense that $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$.

The permutation symbol can alternatively be defined as

$$\epsilon_{ijk} = \det\left([\boldsymbol{e}_i], [\boldsymbol{e}_j], [\boldsymbol{e}_k]\right),$$

where $([\boldsymbol{e}_i], [\boldsymbol{e}_j], [\boldsymbol{e}_k])$ is the 3×3 matrix with columns $[\boldsymbol{e}_i]$, $[\boldsymbol{e}_j]$ and $[\boldsymbol{e}_k]$. In particular, all the properties outlined above for ϵ_{ijk} under index transposition and permutation can be deduced from properties of the determinant.

1.2.3 Frame Identities

Various useful identities can be established between the vectors of a frame $\{\boldsymbol{e}_i\}$ and the Kronecker delta δ_{ij} and the permutation symbol ϵ_{ijk}. In particular, from the definition of δ_{ij} we deduce the identity

$$\boldsymbol{e}_i = \delta_{ij} \boldsymbol{e}_j.$$

That is, $\delta_{1j}\boldsymbol{e}_j = \boldsymbol{e}_1$, $\delta_{2j}\boldsymbol{e}_j = \boldsymbol{e}_2$ and $\delta_{3j}\boldsymbol{e}_j = \boldsymbol{e}_3$. This identity shows that, when δ_{ij} is summed with another quantity, the net effect is a "transfer of index". We call this the **transfer property** of δ_{ij}.

From the definition of ϵ_{ijk} we deduce the identity

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk} \boldsymbol{e}_k,$$

which provides a concise way to express the nine possible vector products between the vectors of a frame, that is, $\boldsymbol{e}_1 \times \boldsymbol{e}_1 = \boldsymbol{0}$, $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$ and so on. A similar identity which can easily be verified is

$$\boldsymbol{e}_i = \tfrac{1}{2}\epsilon_{ijk}\boldsymbol{e}_j \times \boldsymbol{e}_k.$$

1.2.4 Vector Operations in Components

The Kronecker delta and permutation symbols naturally arise when various operations between vectors are expressed in terms of their components in a given frame.

Scalar product. Let $\boldsymbol{a} = a_i\boldsymbol{e}_i$ and $\boldsymbol{b} = b_j\boldsymbol{e}_j$. Then

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b} &= (a_i\boldsymbol{e}_i)\cdot(b_j\boldsymbol{e}_j)\\ &= a_ib_j\ \boldsymbol{e}_i\cdot\boldsymbol{e}_j\\ &= a_ib_j\delta_{ij}\\ &= a_ib_i,\end{aligned}$$

where the last line follows from the transfer property of δ_{ij}, that is, $b_j\delta_{ij} = b_i$. Thus, from the geometrical definition of the scalar product we find $|\boldsymbol{a}|^2 = \boldsymbol{a}\cdot\boldsymbol{a} = a_ia_i$.

Vector product. Let $\boldsymbol{a} = a_i\boldsymbol{e}_i$, $\boldsymbol{b} = b_j\boldsymbol{e}_j$ and $\boldsymbol{d} = d_k\boldsymbol{e}_k$. Then

$$\boldsymbol{a}\times\boldsymbol{b} = a_ib_j\ \boldsymbol{e}_i\times\boldsymbol{e}_j = a_ib_j\epsilon_{ijk}\boldsymbol{e}_k,$$

where the last equality follows from the frame identity $\boldsymbol{e}_i\times\boldsymbol{e}_j = \epsilon_{ijk}\boldsymbol{e}_k$. Thus, if we let $\boldsymbol{d} = \boldsymbol{a}\times\boldsymbol{b}$, then $d_k = a_ib_j\epsilon_{ijk}$.

Triple scalar product. Let $\boldsymbol{a} = a_i\boldsymbol{e}_i$, $\boldsymbol{b} = b_j\boldsymbol{e}_j$ and $\boldsymbol{c} = c_m\boldsymbol{e}_m$. Then

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} &= (a_ib_j\epsilon_{ijk}\boldsymbol{e}_k)\cdot(c_m\boldsymbol{e}_m)\\ &= \epsilon_{ijk}a_ib_jc_m\delta_{km}\\ &= \epsilon_{ijk}a_ib_jc_k.\end{aligned} \tag{1.2}$$

Moreover, by properties of the permutation symbol under cyclic permutation of its indices we find

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = (\boldsymbol{b}\times\boldsymbol{c})\cdot\boldsymbol{a} = (\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}.$$

From elementary vector analysis we recall that

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = \det\left([\boldsymbol{a}],[\boldsymbol{b}],[\boldsymbol{c}]\right), \tag{1.3}$$

where $([\boldsymbol{a}], [\boldsymbol{b}], [\boldsymbol{c}])$ is the 3×3 matrix with columns $[\boldsymbol{a}]$, $[\boldsymbol{b}]$ and $[\boldsymbol{c}]$. When (1.3) is combined with (1.2) we obtain

$$\det([\boldsymbol{a}], [\boldsymbol{b}], [\boldsymbol{c}]) = \epsilon_{ijk} a_i b_j c_k, \tag{1.4}$$

which provides an explicit expression for the determinant in terms of a triple sum.

Triple vector product. Let $\boldsymbol{a} = a_q \boldsymbol{e}_q$, $\boldsymbol{b} = b_i \boldsymbol{e}_i$, $\boldsymbol{c} = c_j \boldsymbol{e}_j$ and $\boldsymbol{d} = d_p \boldsymbol{e}_p$. Then

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (a_q \boldsymbol{e}_q) \times (b_i c_j \epsilon_{ijk} \boldsymbol{e}_k) \\ &= \epsilon_{ijk} a_q b_i c_j \, \boldsymbol{e}_q \times \boldsymbol{e}_k \\ &= \epsilon_{qkp} \epsilon_{ijk} a_q b_i c_j \boldsymbol{e}_p \\ &= \epsilon_{pqk} \epsilon_{ijk} a_q b_i c_j \boldsymbol{e}_p, \end{aligned} \tag{1.5}$$

where the last line follows from the fact that $\epsilon_{qkp} = \epsilon_{pqk}$. Thus, if we let $\boldsymbol{d} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$, then $d_p = \epsilon_{pqk} \epsilon_{ijk} a_q b_i c_j$. Below we show that this expression can be used to derive a fundamental identity for the triple vector product.

Remark: The geometrical definitions of the scalar and triple scalar products imply that these quantities are frame-independent. That is, the scalar and triple scalar products can be computed using components in any coordinate frame, with the same value obtained in each frame. Identifying scalar quantities with this property will be an important theme in the sequel. For other examples see the discussion of the trace, determinant and principal invariants of second-order tensors. □

1.2.5 Epsilon-Delta Identities

By virtue of their definitions in terms of a right-handed orthonormal frame, the permutation symbol and the Kronecker delta satisfy the following identities.

Result 1.1 ***Epsilon-Delta Identities.*** *Let ϵ_{ijk} be the permutation symbol and δ_{ij} the Kronecker delta. Then*

$$\epsilon_{pqs} \epsilon_{nrs} = \delta_{pn} \delta_{qr} - \delta_{pr} \delta_{qn} \quad \textit{and} \quad \epsilon_{pqs} \epsilon_{rqs} = 2\delta_{pr}.$$

□

Proof See Exercise 18. □