Conjugacy in groups of finite Morley rank

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Summary

We survey conjugacy results in groups of finite Morley rank, mixing unipotence, Carter, and Sylow theories in this context.

Introduction

When considering certain classes of groups one might expect conjugacy theorems, and the class of groups of finite Morley rank is not an exception to this. The study of groups of finite Morley rank is mostly motivated by the Algebricity Conjecture, formulated by G. Cherlin and B. Zilber in the late seventies, which postulates that infinite simple groups of this category are isomorphic to algebraic groups over algebraically closed fields. The model-theoretic rank involved appeared in the sixties when M. Morley proved his famous theorem on the categoricity in *any* uncountable cardinal of first order theories categorical in *one* uncountable cardinal [Mor65]. He introduced for that purpose an ordinal valued rank, later shown to be finite by J. Baldwin in the uncountably categorical context [Bal73], and this rank can be seen as an abstract version of the Zariski dimension in algebraic geometry over an algebraically closed field.

In particular, the category of groups of finite Morley rank encapsulates finite groups and algebraic groups over algebraically closed fields. One of the most basic tools for analyzing finite groups is Sylow theory, and in algebraic groups semisimplicity and unipotence theory play a similar role. It is thus not surprising to see these two theories, together with all

† Parts of this work were done while the authors were visiting the Isaac Newton Institute, Cambridge, during the model theory program in the spring 2005.

conjugacy results they suggest, having enormous and close developments in the more abstract category of groups of finite Morley rank. The present paper is intended to give an exhaustive survey on these parallel developments.

In a connected linear algebraic group, the centralizers of maximal tori are conjugate and cover the group generically. In the category of groups of finite Morley rank, these Cartan subgroups are best approximated by *Carter* subgroups, which are defined merely by the outstanding properties of being definable, connected, nilpotent, and of finite index in their normalizers. The main feature of Carter subgroups is their existence in any group of finite Morley rank. They constitute, together with all relevant approximations of semisimplicity and unipotence, the core of our preoccupations in this paper.

Sylow theory, as the study of maximal p-subgroups, is well understood for any p in *solvable* groups of finite Morley rank, and in any group of finite Morley rank for the prime $p = 2$. The second point is the key for a classification program of simple groups of finite Morley rank, suggested by A. Borovik and based on the architecture of the Classification of the Finite Simple Groups. In this process, some specific developments have naturally been needed for groups of finite Morley rank. In this context there is a priori no Jordan decomposition as in the linear algebraic context, and hence no nice distinction between semisimple and unipotent elements. The situation is furthermore enormously complicated by some so-called *bad* fields, as we will see in §1.7. Nevertheless, the finiteness of the Morley rank has allowed J. Burdges to develop a graduated notion of unipotence in this general context. This graduated notion of unipotence leads naturally to a new kind of Sylow theory, not related to torsion elements directly, but rather to the *unipotence degree* of the subgroups involved. In finite groups the study of Carter subgroups mostly boils down to Sylow theory; in groups of finite Morley rank this is replaced by this new kind of Sylow theory.

More precisely, we deal here with $\tilde{p}$-groups, where $\tilde{p} = (p, r)$ and p should be understood as the usual prime, or ∞ when dealing with elements of infinite order or merely divisible groups (which is more or less the same up to saturation). In this theory the unipotence degree r measures simultaneously how much a $\tilde{p}$-group can act on, and be acted upon by, another such group. Our $\tilde{p}$-groups are connected and nilpotent by definition and can really be thought of as the p-groups from finite group theory, incorporating the important unipotence degree parameter in our context. They are of three types depending on the value of $\tilde{p}$, listed below by increasing unipotence degree.

- $(\infty, 0)$-groups, or (abelian) "decent tori",
- (∞, r)-groups, with $0 < r < \infty$, or "nilpotent Burdges' $U_{\tilde{p}}$-groups",
- (p, ∞)-groups, with p prime, or (nilpotent) "p-unipotent groups".

This will be explained in §2. In particular, we will see in §2.4 that these $\tilde{p}$-groups cover in some sense all the "basic" connected groups which can occur in our context.

Imposing maximality on these $\tilde{p}$-groups leads naturally to a notion of Sylow theory, reminiscent of that of finite group theory. These new Sylow $\tilde{p}$-subgroups allow one to show the existence of Carter subgroups in any group of finite Morley rank, and hence to have a good approximation of Cartan subgroups of an algebraic group in any case. The natural question arising then is that of their conjugacy. This remains an open problem in general, but we will see in §3 that conjugacy of Carter subgroups is known in two important cases: under a generosity assumption on the one hand, and in solvable groups on the other. We say that a definable subgroup is *generous* if its conjugates cover the ambient group generically. Generosity appeared over the years to be a weak form of conjugacy, and this is confirmed for Carter subgroups also. More precisely, we will see in §3.3 below that an arbitrary group of finite Morley rank contains at most one conjugacy class of generous Carter subgroups. Using this conjugacy result by generosity, we rework then the theory of Carter subgroups in connected solvable groups of finite Morley rank, which was well developed before the unipotence theory mentioned above came into play.

In the present paper we are mostly concerned with conjugacy of certain *connected* subgroups, except in the parenthetical §6.5 which deals with nonnecessarily connected solvable groups of finite Morley rank. In §3.8 we will also compare the theory of Carter subgroups in groups of finite Morley rank, which relies heavily on connectedness, to its analog in finite group theory, where of course connectedness has no exact analog. Concerning the conjugacy of certain connected subgroups of groups of finite Morley rank, the most challenging conjectures are probably the three following.

Conjugacy Conjectures *In any group of finite Morley rank,*

1.12 *Borel subgroups are conjugate,*

2.6 *Sylow $\tilde{p}$-subgroups are conjugate,*

3.1 *Carter subgroups are conjugate.*

We will never consider Conjecture 1.12 here, but we will see that it is stronger than the two others, as we will see the conjugacy of Carter subgroups and of Sylow $\tilde{p}$-subgroups in connected solvable groups. As visible already, solvable groups satisfy many conjugacy theorems. This is merely because they mesh perfectly well with induction arguments, and for this reason the majority of results surveyed here concern solvable groups of finite Morley rank.

For finite solvable groups, *formation theory* is a general and powerful framework for analyzing the interplay between Sylow subgroups, Carter subgroups, and conjugacy. In §4 below we develop, with new results, a very general subformation theory designed for connected solvable groups of finite Morley rank. This theory encapsulates Carter subgroups and several generalizations of Sylow $\tilde{p}$-subgroups in connected solvable groups of finite Morley rank. All expectable conjugacy theorems are derived in §5 below.

To summarize, the architecture of this paper is as follows. The first section concerns preliminary developments on groups of finite Morley rank, with an emphasis on classical, i.e. involving torsion elements, Sylow theory in solvable groups. Then we develop in §2 the theory of semisimplicity and unipotence. In §3 we consider Carter subgroups, with their existence in general and their conjugacy in two important cases. Then in §4 and §5 we are concerned with subformation theory. In §4 a general and quite formal subformation theory is developed, and in §5 we give applications, with conjugacy and structural theorems in connected solvable groups. Finally, §6 deals with additional structural results in solvable groups of finite Morley rank, which are of a slightly different nature but of a certain interest. To conclude, we give in §7 a few examples of applications of this theory beyond solvable groups.

All developments and related notions presented here are well understood under a *linearity* assumption, thanks to the work of Y. Mustafin [Mus04]. Here we work with no linearity assumption and refer to that paper for linear groups.

1 Preliminary developments

We start with some early developments on groups of finite Morley rank.

1.1 Borovik-Poizat axioms

We consider groups $\langle G, \cdot, ^{-1}, 1, \cdots \rangle$ from a model theoretic point of view and think in G^{eq} throughout. They may carry additional structure, and this is an important issue in this context. For example, a group definable in another may carry extra structure not definable in its own pure group structure. We say that a group is *ranked* if there is a function "rk", assigning to each nonempty definable set an integer (its *rank*, or dimension) and satisfying the following axioms for every definable sets A and B:

Definition: $\mathrm{rk}\,(A) \geq n+1$ if and only if A contains infinitely many pairwise disjoint subsets A_i such that $\mathrm{rk}\,(A_i) \geq n$.

Definability: For every uniformly definable family A_b of subsets of A, with b varying in B, the set of elements $b \in B$ such that A_b is of given rank n is a definable subset of B.

Finite sets: For every uniformly definable family A_b of subsets of A, with b varying in B, there is a uniform bound on the cardinals of the finite sets A_b.

In an arbitrary structure, the existence of such a rank implies superstability [BC02]. In a group theoretic context this is equivalent to the finiteness of Morley rank [Poi87], implying in particular the additivity of the rank. We rather tend to work with these purely combinatorical axioms and the book [BN94a] develops all the theory from them.

1.2 Connectedness

If $X \subseteq Y$ are two definable sets in a group of finite Morley rank, we say that X is *generic* in Y if $\mathrm{rk}\,(X) = \mathrm{rk}\,(Y)$. Each definable set has a finite (Morley) *degree*, the maximal number of disjoint generic subsets. It follows easily that groups of finite Morley rank satisfy the Descending Chain Condition on definable subgroups. In particular, such a group G has a smallest definable subgroup of finite index, the intersection of all of them, its *connected component* denoted by G°. It is of course a definably characteristic subgroup, and G is said to be *connected* if $G = G^\circ$. The main property of connected groups can be stated as follows.

Lemma 1.1 [Che79] *A group of finite Morley rank is connected if and only if it has Morley degree one.*

We are going to deal essentially with connected groups. Most of the time, connectedness allows one to avoid all complications from finite combinatorics. For example, the following corollary of Lemma 1.1 has a striking application in §3.3 below.

Corollary 1.2 *A connected group of finite Morley rank acting definably on a finite set fixes it pointwise.*

By the Descending Chain Condition on definable subgroups again, each (nonnecessarily definable) subset X of a group G of finite Morley rank is contained in a smallest definable subgroup $d(X)$, its *definable closure*, which can be seen as a sharper form of the Zariski closure in the algebraic context. This allows one to define the *generalized* connected component of X as $X^\circ = d(X)^\circ \cap X$. If X is a subgroup of G, one sees easily that X° is still, though not necessarily definable, a normal subgroup of finite index in X, and again one says that X is *connected* if $X = X^\circ$. This generalized connected component is particularly relevant for Sylow theory in the classical sense of the study of torsion subgroups. For example, the torsion subgroup of the multiplicative group $\mathbb{C}^\times$ of the complex numbers is not first-order definable in the pure field structure.

For the sake of future arguments, we include here some corollaries of Zilber's theorem on indecomposable sets and connectedness.

Theorem 1.3 ([Zil77]; [BN94a, Corollary 5.29]) *Let G be a group of finite Morley rank. Then:*

a. Any family of definable connected subgroups of G generates a definable connected subgroup of G.
b. If H is a definable connected subgroup of G and X any subset, then the commutator subgroup $[H, X]$ is a definable connected subgroup of G.

1.3 Classical Sylow theory

If p is a prime, a *p-torus* is a divisible abelian p-subgroup of a group of finite Morley rank. By abelian group theory, such a subgroup is a direct product of copies of the quasicylic Prüfer p-group $\mathbb{Z}_{p^\infty}$. In the finite Morley rank context, the number of copies must be *finite* [BP90], and is called the *Prüfer p-rank*. Typically, if K is an algebraically closed field of characteristic different from p, then the n-dimensional torus $K^\times \times \cdots \times K^\times$ contains a p-torus of Prüfer p-rank n.

At the opposite of p-tori, which are of unbounded exponent and not necessarily definable, *p-unipotent* subgroups are the definable connected nilpotent p-subgroups of bounded exponent of groups of finite Morley rank. Notice nilpotence in our definition. A typical example of a p-unipotent group is the group of strictly upper triangular matrices of the general linear group $\mathrm{GL}_n(K)$, with K an algebraically closed field of characteristic p.

The following result describes locally finite p-subgroups of groups of finite Morley rank, mostly in terms of a p-torus and of a p-unipotent subgroup.

Theorem 1.4 [BP90] *Let P be a locally finite p-subgroup, p prime, of a group of finite Morley rank. Then $P^\circ = T * U$ is a central product, with finite intersection, of a p-torus T and a p-unipotent subgroup U.*

It is well known that torsion subgroups of solvable groups are locally finite, and thus the preceding theorem applies in particular to any p-subgroup of a solvable group of finite Morley rank. As usual, one defines *Sylow* p-subgroups as maximal p-subgroups, or, equivalently in a solvable context, as maximal locally finite p-subgroups. The conjugacy of Sylow p-subgroups is not known in general, except in a solvable context or for the prime $p = 2$. Indeed, the singularity of the prime $p = 2$ yields an absolute control of 2-subgroups.

Theorem 1.5 [BP90] *In any group of finite Morley rank, maximal* 2*-subgroups are locally finite and conjugate.*

Theorem 1.5 is the origin of many arguments in the presence of non-trivial 2-elements. Here we are going to concentrate on aspects not depending on such a presence, hence on other primes and even (mostly, indeed) elements of infinite order.

As alluded to already, there is a conjugacy theorem for Sylow p-subgroups in a solvable context. This is indeed true for a larger class of torsion subgroups. If π is a set of primes, then a *Hall* π-subgroup of a solvable group G of finite Morley rank is a maximal π-subgroup of G.

Theorem 1.6 [ACCN98] *In any solvable group of finite Morley rank, Hall π-subgroups are conjugate for any set π of primes.*

There is an analog of the structural Theorem 1.4 for Hall π-subgroups.

Theorem 1.7 [Fré00c, Proposition 4.22] *Let G be a solvable group of finite Morley rank and R a Hall π-subgroup of G. Then $R^\circ = TU$ where $U \trianglelefteq G$ is a definable connected subgroup of bounded exponent and T an abelian divisible subgroup.*

Also, as in the algebraic context, Hall π-subgroups of *connected* solvable groups of finite Morley rank are connected.

Theorem 1.8 ([Fré00b, Corollaire 7.15], see also [BN92]) *Let G be a connected solvable group of finite Morley rank. Then Hall π-subgroups of G are connected.*

Finally, there are results of a Schur-Zassenhaus type, due to A. Borovik and A. Nesin.

Theorem 1.9 [BN92, BN94b] *Let G be a solvable group of finite Morley rank and H a normal Hall π-subgroup of G. Then:*

a. H has a complement in G.
b. If H is of bounded exponent, then any subgroup K of G with $K \cap H = 1$ is contained in a complement of H in G, and the complements of H in G are definable and conjugate.

1.4 Generalized Hall π-subgroups

Of course, the preceding theorems depend heavily on the presence of torsion elements. To deal with infinite groups of finite Morley rank, one would also like some kind of similar theory for elements of infinite order, at least in solvable groups again. An attempt in this direction is taken in [Fré00c], leading to the following definition. Denoting by $\mathcal{P}$ the set of all primes together with the ∞ symbol, we consider arbitrary subsets π of $\mathcal{P}$ and $\pi^\perp = \mathcal{P} \setminus \pi$. If G is a solvable group of finite Morley rank and R a subgroup of G, we say that:

- An element $x \in G$ is a π-*element* if, for every $p \in \pi^\perp$, $d(x)$ has no elements of order p.
- R is a π-*subgroup* if each $x \in R$ is a π-element.
- R is a *Hall* π-subgroup of G if R is a maximal π-subgroup of G.

Of course, this definition coincides with the usual one if π consists of finite primes only. The main feature of this definition allowing the infinite prime is that Theorems 1.6 and 1.7 on conjugacy and structure are preserved.

Theorem 1.10 [Fré00c, Théorème 4.18, Proposition 4.22] *Let G be a solvable group of finite Morley rank and π any subset of $\mathcal{P}$. Then:*

a. Hall π-subgroups are conjugate.
b. For any Hall π-subgroup R of G, $R^\circ = UBD$ where $U \trianglelefteq G$ is a definable torsion-free subgroup, $B \trianglelefteq G$ a definable connected subgroup of bounded exponent and D a divisible nilpotent subgroup. Moreover, R is locally closed in the sense of §6.5 below.

If the ambient group G is connected, then it is also shown in [Fré00c] that its Hall π-subgroups in this generalized sense are connected.

1.5 Borel subgroups

As visible already, conjugacy theorems are particularly abundant in solvable groups of finite Morley rank. The following result links a given group of finite Morley rank to its solvable subgroups.

Theorem 1.11 ([Fré00a, Corollaire 3.4.4], see also [ACCN98]) *Every locally solvable subgroup of a group of finite Morley rank is solvable.*

In general, we are mostly concerned with definable connected subgroups. A *Borel* subgroup of a group G of finite Morley rank is a maximal definable connected solvable subgroup of G. The following very strong conjecture is widely open.

Conjecture 1.12 *In any group of finite Morley rank, Borel subgroups are conjugate.*

Conjecture 1.12 covers all natural conjugacy conjectures which are formulated here about connected subgroups of groups of finite Morley rank. For example, we will see that it is stronger than both Conjectures 3.1 and 2.6 below, by Theorems 3.11 and 5.10 respectively. In particular, the class of connected solvable groups of finite Morley rank is very well understood, and this is extremely relevant as the analysis of an arbitrary group of finite Morley rank is often done with its Borel subgroups.

1.6 Actions

If X and Y are two definable subgroups of a group of finite Morley rank, then X is *Y-minimal* if it is infinite, normalized by Y, and minimal with respect to these properties. In a solvable context, the study of a serious action of a group of finite Morley rank on another most of the time boils down to the following crucial theorem, which gives in many cases an interpretable field.

Zilber's Field Theorem (cf. [BN94a, Theorem 9.1]) *Let $G = U \rtimes T$ be a group of finite Morley rank, with U and T infinite abelian definable subgroups, $C_T(U) = 1$ and U T-minimal. Then G interprets an algebraically closed field K with U definably isomorphic to K_+, T definably isomorphic to a definable subgroup T_1 of $K^\times$, and*

$$U \rtimes T \simeq K_+ \rtimes T_1 = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} : t \in T_1 \ , u \in K_+ \right\}.$$

The *Fitting* subgroup of a group G of finite Morley rank is its maximal normal definable nilpotent subgroup. It is well defined and the unique maximal normal nilpotent subgroup of G [Nes91]. If B is a Borel subgroup of a linear algebraic group over an algebraically closed field, then $B = U \rtimes T$ where U is the maximal unipotent subgroup (strictly upper triangular matrices if B is the standard Borel subgroup) and T is a maximal torus of B (diagonal matrices). If the ambient group is simple, then $U = F(B)$ and, hence, Fitting subgroups are usually a good first approximation of the unipotent radical in the finite Morley rank context. In general, it is not known whether a connected solvable group B of finite Morley rank splits as $F(B) \rtimes T$ for some complement T. But any Carter subgroup Q of B satisfies $B = F(B)Q$ by Corollary 3.13 below, and hence provides a good approximation of maximal tori in this context.

For future references, we record here miscellaneous results around connected solvable groups of finite Morley rank.

Theorem 1.13 ([Nes90]; [BN94a, Ex. 5 p. 98]) *Let G be a connected solvable group of finite Morley rank. Then:*

a. $G/F^\circ(G)$ is divisible abelian. In particular G' is nilpotent.
b. If A is G-minimal in G, then $A \leq Z(F^\circ(G))$

1.7 Fields

An infinite field of finite Morley rank is always algebraically closed [Mac71]. If it is involved in some action, one might become extremely concerned with the definable subgroups of its multiplicative or additive