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AN INTRODUCTION TO MANY-VALUED AND FUZZY LOGIC

This volume is an accessible introduction to the subject of many-valued and fuzzy logic suitable for use in relevant advanced undergraduate and graduate courses. The text opens with a discussion of the philosophical issues that give rise to fuzzy logic—problems arising from vague language—and returns to those issues as logical systems are presented. For historical and pedagogical reasons, three-valued logical systems are presented as useful intermediate systems for studying the principles and theory behind fuzzy logic. The major fuzzy logical systems—Łukasiewicz, Gödel, and product logics—are then presented as generalizations of three-valued systems that successfully address the problems of vagueness. Semantic and axiomatic systems for three-valued and fuzzy logics are examined along with an introduction to the algebras characteristic of those systems. A clear presentation of technical concepts, this book includes exercises throughout the text that pose straightforward problems, ask students to continue proofs begun in the text, and engage them in the comparison of logical systems.

Merrie Bergmann is an emerita professor of computer science at Smith College. She is the coauthor, with James Moor and Jack Nelson, of *The Logic Book*.

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AN INTRODUCTION TO

Many-Valued and Fuzzy Logic

SEMANTICS, ALGEBRAS, AND DERIVATION SYSTEMS

Merrie Bergmann

Emerita, Smith College

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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

79 Anson Road, #06-04/06, Singapore 079906

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www.cambridge.org
Information on this title: www.cambridge.org/9780521707572

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First published 2008

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging in Publication data
Bergmann,Merrie.
An introduction to many-valued and fuzzy logic : semantics, algebras, and derivation systems /Merrie Bergmann.
p. cm.
Includes bibliographical references and index.
ISBN 978-0-521-88128-9 (hardback) – ISBN 978-0-521-70757-2
1. Fuzzy logic. 2. Many-valued logic. I. Title.
QA9.64.B47 2008
511.3´13 – dc22 2007007725

ISBN 978-0-521-88128-9 Hardback
ISBN 978-0-521-70757-2 Paperback

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To my husband, Michael Thorpe, with love,

for his understanding and support during this project

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Preface

Formal fuzzy logic has developed into an extensive, rigorous, and exciting discipline since it was first proposed by Joseph Goguen and Lotfi Zadeh in the midtwentieth century, and it is a wonderful topic for introducing students to the richness and fascination of formal logic and the philosophy thereof. This textbook grew out of an interdisciplinary course on fuzzy logic that I've taught at Smith College, a course that attracts philosophy, computer science, and mathematics majors. I taught the course for several years with only a course reader because the few existing texts devoted to fuzzy logic were too advanced for my undergraduate audience (and probably for some graduate audiences as well). Finally, after writing voluminous supplements for the course, I decided to write an accessible introductory textbook on many-valued and fuzzy logic. It is my hope that after working through this textbook, students will have the necessary background to tackle more advanced texts, such as Gottwald (2001), Hájek (1998b), and Novák, Perfilieva, and Močkoř (1999), along with the rest of the vast fuzzy literature.

This book opens with a discussion of the philosophical issues that give rise to fuzzy logic—problems and paradoxes arising from vague language—and returns to those issues as new logical systems are presented. There is a two-chapter review of classical logic to familiarize students and instructors with my terminology and notation, and to introduce formal logic to those who have no prior background. Three-valued logical systems are introduced as candidate logics for vagueness, ultimately to be rejected but interesting in their own right and serving as useful intermediate systems for studying the principles and theory that guide fuzzy logics. The major fuzzy logical systems—Łukasiewicz, Gödel, and product logics—are then presented as generalizations of three-valued systems, generalizations that fully address the problems of vagueness. The text ends with two chapters introducing further directions for study: extensions of basic fuzzy systems and definitions of fuzzy membership functions.

Throughout, I have included both semantic and axiomatic systems, along with introductions to the algebras characteristic of those systems. Many texts that have a chapter or so on fuzzy logic restrict their attention to semantics, but much of the interest of fuzzy logic lies in the rich axiomatic systems developed by Jan Pavelka and in the insights garnered from studying the algebras for these systems.

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I've used semantic concepts that aren't featured in standard presentations of fuzzy logic, specifically, the concepts of degree-validity and n-degree-validity (these concepts were proposed in Machina (1976)). Degree-validity occurs when an argument's conclusion is guaranteed to be at least as true as the least true premise and is an obvious generalization of classical validity. N-degree-validity measures the slippage of truth going from premises to conclusion: how much less true than the premises can the conclusion of an argument be? The latter concept is particularly useful in analyzing Sorites arguments, and in comparing the performance of the three major fuzzy logical systems with respect to these arguments.

There are exercises throughout the text. Some pose straightforward problems for the student to solve, but many exercises also ask students to continue proofs begun in the text, to prove results analogous to those in the text, and to compare the various logical systems that are presented.

This textbook can be used as a complete basis for an introductory course on formal many-valued and fuzzy logics, at either the upper-level undergraduate or the graduate level, and it can also be used as a supplementary text in a variety of courses. There is considerable flexibility in either case. The truth-valued semantic chapters are independent of the algebraic and axiomatic ones, so that either of the latter may be skipped. Except for Section 13.3 of Chapter 13, the axiomatic chapters are also independent of the algebraic ones, and an instructor who chooses to skip the algebraic material can simply ignore the latter part of 13.3. Finally, Łukasiewicz fuzzy logic is presented independently of Gödel and product fuzzy logics, thus allowing an instructor to focus solely on the former.

I am indebted to my students at Smith College for making this course such a pleasure to teach, and for the many questions and comments that have informed my presentations throughout the text. Joseph Goguen and Petr Hájek, the two men whose work most largely generated my own appreciation of fuzzy logic, generously answered questions that I e-mailed as I was writing the text. It was with great sadness that I learned of Professor Goguen's passing at the age of sixty-five last summer; fuzzy logic as we know it owes much to his pioneering work.

I also thank my colleague Michael Albertson for a helpful analytic suggestion that I used in Chapter 14, and two anonymous reviewers of several chapters for their careful reading and thoughtful suggestions. Any inelegance or errors remain my responsibility alone. Finally, I thank Smith College for generous sabbatical release time.

Merrie Bergmann
August 2007

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