Volume 1

Survey Articles

Cambridge University Press
978-0-521-70174-7 - Algebraic Cycles and Motives: Volume 1
Edited by Jan Nagel and Chris Peters
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The Motivic Vanishing Cycles and the Conservation Conjecture

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To Jacob Murre for his 75th birthday

1.1 Introduction

Let X be a noetherian scheme. Following Morel and Voevodsky (see [24], [25], [28], [33] and [37]), one can associate to X the motivic stable homotopy category $\mathbf{SH}(X)$. Objects of $\mathbf{SH}(X)$ are T-spectra of simplicial sheaves on the smooth Nisnevich site $(\mathrm{Sm}/X)_{\mathrm{Nis}}$, where T is the pointed quotient sheaf $\mathbb{A}^1_X/\mathbb{G}\mathrm{m}_X$. As in topology, $\mathbf{SH}(X)$ is triangulated in a natural way. There is also a tensor product $-\otimes_X-$ and an "internal hom": $\underline{\mathrm{Hom}}_X$ on $\mathbf{SH}(X)$ (see [20] and [33]). Given a morphism $f : X \longrightarrow Y$ of noetherian schemes, there is a pair of adjoint functors (f^*, f_*) between $\mathbf{SH}(X)$ and $\mathbf{SH}(Y)$. When f is quasi-projective, one can extend the pair (f^*, f_*) to a quadruple $(f^*, f_*, f_!, f^!)$ (see [3] and [8]). In particular we have for $\mathbf{SH}(-)$ the full package of the Grothendieck six operators. It is then natural to ask if we have also the seventh one, that is, if we have a vanishing cycle formalism (analogous to the one in the étale case, developed in [9] and [10]).

In the third chapter of our PhD thesis [3], we have constructed a vanishing cycles formalism for motives. The goal of this paper is to give a detailed account of that construction, to put it in a historical perspective and to discuss some applications and conjectures. In some sense, it is complementary to [3] as it gives a quick introduction to the theory with emphasis on motivations rather than a systematic treatment. The reader will not find all the details here: some proofs will be omitted or quickly sketched, some results will be stated with some additional assumptions (indeed we will be mainly interested in motives with rational coefficients over characteristic zero schemes).

For the full details of the theory, one should consult [3]. Let us mention also that M. Spitzweck has a theory of limiting motives which is closely related to our motivic vanishing cycles formalism. For more information, see [35].

The paper is organized as follows. First we recall the classical pictures: the étale and the Hodge cases. Although this is not achieved here, these classical constructions should be in a precise sense realizations of our motivic construction. In section 1.3 we introduce the notion of a specialization system which encodes some formal properties of the family of nearby cycles functors. We state without proofs some important theorems about specialization systems obtained in [3]. In section 1.4, we give our main construction and prove motivic analogues of some well-known classical results about nearby cycles functors: constructibility, commutation with tensor product and duality, etc. We also construct a monodromy operator on the unipotent part of the nearby cycles which is shown to be nilpotent. Finally, we propose a conservation conjecture which is weaker than the conservation of the classical realizations but strong enough to imply the Schur finiteness of constructible motives‡.

In the literature, the functors Ψ_f have two names: they are called "nearby cycles functors" or "vanishing cycles functors". Here we choose to call them the nearby cycles functors. The properties of these functors form what we call the vanishing cycles formalism (as in [9] and [10]).

1.2 The classical pictures

We briefly recall the construction of the nearby cycles functors $\mathsf{R}\Psi_f$ in étale cohomology. We then explain a construction of Rapoport and Zink which was the starting point of our definition of Ψ_f in the motivic context. After that we shall recall some facts about limits of variations of Hodge structures. A very nice exposition of these matters can be found in [15].

1.2.1 The vanishing cycles formalism in étale cohomology

Let us fix a prime number ℓ and a finite commutative ring Λ such that $\ell^\nu.\Lambda = 0$ for ν large enough. When dealing with étale cohomology, we shall always assume that ℓ is invertible on our schemes. For a reasonable scheme V, we denote by $\mathsf{D}^+(V, \Lambda)$ the derived category of bounded below complexes of étale sheaves on V with values in Λ-modules.

‡ *Constructible motives* means *geometric motives* of [40]. They are also the compact objects in the sense Neeman [30] (see remark 1.3.3).

Let S be the spectrum of a strictly henselian DVR (discrete valuation ring). We denote by η the generic point of S and by s the closed point:

$$\eta \xrightarrow{j} S \xleftarrow{i} s.$$

We also fix a separable closure $\bar{\eta}$ of the point η. From the point of view of étale cohomology, the scheme S plays the role of a small disk so that η is a punctured small disk and $\bar{\eta}$ is a universal cover of that punctured disk. We will also need the normalization $\bar{S}$ of S in $\bar{\eta}$:

$$\bar{\eta} \xrightarrow{\bar{j}} \bar{S} \xleftarrow{\bar{i}} s.$$

Now let $f : X \longrightarrow S$ be a finite type S-scheme. We consider the commutative diagram with cartesian squares

$$\begin{array}{ccccc} X_\eta & \xrightarrow{j} & X & \xleftarrow{i} & X_s. \\ {\scriptstyle f_\eta}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_s} \\ \eta & \xrightarrow{j} & S & \xleftarrow{i} & s \end{array}$$

Following Grothendieck (see [10]), we look also at the diagram

$$\begin{array}{ccccc} X_{\bar{\eta}} & \xrightarrow{\bar{j}} & \bar{X} & \xleftarrow{\bar{i}} & X_s \\ {\scriptstyle f_{\bar{\eta}}}\downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f_s} \\ \bar{\eta} & \xrightarrow{\bar{j}} & \bar{S} & \xleftarrow{\bar{i}} & s \end{array}$$

obtained in the same way by base-changing the morphism f. (This is what we will call the "Grothendieck trick"). We define then the triangulated functor:

$$\mathsf{R}\Psi_f : \mathsf{D}^+(X_\eta, \Lambda) \longrightarrow \mathsf{D}^+(X_s, \Lambda)$$

by the formula: $\mathsf{R}\Psi_f(A) = \bar{i}^*\mathsf{R}\bar{j}_*(A_{X_{\bar{\eta}}})$ for $A \in \mathsf{D}^+(X_\eta, \Lambda)$. By construction, the functor $\mathsf{R}\Psi_f$ comes with an action of the Galois group of $\bar{\eta}/\eta$, but we will not explicitly use this here. The basic properties of these functors concern the relation between $\mathsf{R}\Psi_g$ and $\mathsf{R}\Psi_{g\circ h}$ (see [9]):

Proposition 1.2.1. *Let $g : Y \longrightarrow S$ be an S-scheme and suppose given an S-morphism $h : X \longrightarrow Y$ such that $f = g \circ h$. We form the commutative diagram*

$$\begin{array}{ccccc} X_\eta & \xrightarrow{j} & X & \xleftarrow{i} & X_s \\ {\scriptstyle h_\eta}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_s} \\ Y_\eta & \xrightarrow{j} & Y & \xleftarrow{i} & Y_s. \end{array}$$

There exist natural transformations of functors

- $\alpha_h : h_s^* \mathsf{R}\Psi_g \longrightarrow \mathsf{R}\Psi_f h_\eta^*$,
- $\beta_h : \mathsf{R}\Psi_g \mathsf{R}h_{\eta *} \longrightarrow \mathsf{R}h_{s*}\mathsf{R}\Psi_f$.

Furthermore, α_h is an isomorphism when h is smooth and β_h is an isomorphism when h is proper.

The most important case, is maybe when $g = \mathrm{id}_S$ and $f = h$. Using the easy fact that $\mathsf{R}\Psi_{\mathrm{id}_S}\Lambda = \Lambda$, we get that:

- $\mathsf{R}\Psi_f \Lambda = \Lambda$ if f is smooth,
- $\mathsf{R}\Psi_{\mathrm{id}_S}\mathsf{R}f_{\eta *}\Lambda = \mathsf{R}f_{s*}\mathsf{R}\Psi_f\Lambda$ if f is proper.

The last formula can be rewritten in the following more expressive way: $\mathrm{H}^*_{\text{ét}}(X_{\bar{\eta}}, \Lambda) = \mathrm{H}^*_{\text{ét}}(X_s, \mathsf{R}\Psi_f\Lambda)$. In concrete terms, this means that for a proper S-scheme X, the étale cohomology of the constant sheaf on the generic geometric fiber $X_{\bar{\eta}}$ is isomorphic to the étale cohomology of the special fiber X_s with value in the complex of nearby cycles $\mathsf{R}\Psi_f\Lambda$. This is a very useful fact, because usually the scheme X_s is simpler than $X_{\bar{\eta}}$ and the complex $\mathsf{R}\Psi_f\Lambda$ can often be computed using local methods.

1.2.2 The Rapoport-Zink construction

We keep the notations of the previous paragraph. We now suppose that X is a semi-stable S-scheme i.e. locally for the étale topology X is isomorphic to the standard scheme $S[t_1, \dots, t_n]/(t_1 \dots t_r - \pi)$ where π is a uniformizer of S and $r \leq n$ are positive integers. In [32], Rapoport and Zink constructed an important model of the complex $\mathsf{R}\Psi_f(\Lambda)$. Their construction is based on the following two facts:

- There exists a canonical arrow $\theta : \Lambda_\eta \longrightarrow \Lambda_\eta(1)[1]$ in $\mathsf{D}^+(\eta, \Lambda)$ called the fundamental class with the property that the composition $\theta \circ \theta$ is zero,
- The morphism $\theta : i^*\mathsf{R}j_*\Lambda \longrightarrow i^*\mathsf{R}j_*\Lambda(1)[1]$ in $\mathsf{D}^+(X_s, \Lambda)$ has a representative on the level of complexes $\underline{\theta} : \mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet(1)[1]$ such that the composition
$$\mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet(1)[1] \longrightarrow \mathcal{M}^\bullet(2)[2]$$
is zero as a map of complexes.

Therefore we obtain a double complex

$$\mathcal{RZ}^{\bullet,\bullet} = [\cdots \to 0 \to \mathcal{M}^\bullet(1)[1] \to \mathcal{M}^\bullet(2)[2] \to \mathcal{M}^\bullet(3)[3] \\ \to \cdots \to \mathcal{M}^\bullet(n)[n] \to \dots]$$

where the complex $\mathcal{M}^\bullet(1)[1]$ is placed in degree zero. Furthermore, following Rapoport and Zink, we get a map $\mathsf{R}\Psi_f\Lambda \longrightarrow \mathsf{Tot}(\mathcal{RZ}^{\bullet,\bullet})$ which is an isomorphism in $\mathsf{D}^+(X_s, \Lambda)$ (see [32] for more details). Here $\mathsf{Tot}(-)$ means the simple complex associated to a double complex. In particular, Rapoport and Zink's result says that the nearby cycles complex $\mathsf{R}\Psi_f\Lambda$ can be constructed using two ingredients:

- The complex $i^*\mathsf{R}j_*\Lambda$,
- The fundamental class θ.

Our construction of the nearby cycles functor in the motivic context is inspired by this fact. Indeed, the above ingredients are motivic (see 1.4.1 for a definition of the motivic fundamental class). We will construct in paragraph 1.4.2 a motivic analogue of $\mathcal{RZ}^{\bullet,\bullet}$ based on these two motivic ingredients and then define the (unipotent) "motivic nearby cycles" to be the associated total motive. In fact, for technical reasons, we preferred to use a motivic analogue of the dual version of $\mathcal{RZ}^{\bullet,\bullet}$. By the dual of the Rapoport-Zink complex, we mean the bicomplex

$$\mathcal{Q}^{\bullet,\bullet} = [\cdots \to \mathcal{M}^\bullet(-n)[-n] \to \cdots \to \mathcal{M}^\bullet(-1)[-1] \to \mathcal{M}^\bullet \to 0 \to \dots]$$

where the complex $\mathcal{M}^\bullet$ is placed in degree zero. It is true that by passing to the total complex, the double complex $\mathcal{Q}^{\bullet,\bullet}$ gives in the same way as $\mathcal{RZ}^{\bullet,\bullet}$ the nearby cycles complex.

1.2.3 The limit of a variation of Hodge structures

Let D be a small analytic disk, 0 a point of D and $D^\star = D - 0$. Let $f : X^\star \longrightarrow D^\star$ be an analytic family of smooth projective varieties. For $t \in D^\star$, we denote by X_t the fiber $f^{-1}(t)$ of f. For any integer q, the local system $\mathsf{R}^q f_*\mathbb{C} = (\mathsf{R}^q f_*\mathbb{Z}) \otimes \mathbb{C}$ on $D^\star$ with fibers $(\mathsf{R}^q f_*\mathbb{C})_t = \mathrm{H}^q(X_t, \mathbb{C})$ is the sheaf of horizontal sections of the Gauss-Manin connection ∇ on $\mathsf{R}^q f_*\Omega^{\cdot}_{X^\star/D^\star}$. The decreasing filtration F^k on the de Rham complex $\Omega^{\cdot}_{X^\star/D^\star}$ given by

$$F^k\Omega^{\cdot}_{X^\star/D^\star} = [0 \to \dots 0 \to \Omega^k_{X^\star/D^\star} \to \cdots \to \Omega^n_{X^\star/D^\star}]$$

induces a filtration $F^k\mathsf{R}^q f_*\Omega^{\cdot}_{X^\star/D^\star}$ by locally free $\mathcal{O}_{D^\star}$-submodules on $\mathsf{R}^q f_*\Omega^{\cdot}_{X^\star/D^\star}$.

For any $t \in D^\star$, we get by applying the tensor product $- \otimes_{\mathcal{O}_{D^\star}} \mathbb{C}(t)$ a filtration F^k on $\mathrm{H}^q(X_t, \mathbb{C})$ which is the Hodge filtration. The data:

- The local system $\mathsf{R}^q f_* \mathbb{Z}$,
- The $\mathcal{O}_{D^\star}$-module $(\mathsf{R}^q f_* \mathbb{Z}) \otimes \mathcal{O}_{D^\star} = \mathsf{R}^q f_* \Omega^{\cdot}_{X^\star/D^\star}$ together with the Gauss-Manin connexion,
- The filtration F^k on $(\mathsf{R}^q f_* \mathbb{Z}) \otimes \mathcal{O}_{D^\star}$

satisfy the Griffiths transversality condition and are called a Variation of (pure) Hodge Structures.

Let us suppose for simplicity that f extends to a semi-stable proper analytic morphism: $X \longrightarrow D$. We denote by $\omega^{\cdot}_{X/D}$ the relative de Rham complex with logarithmic poles on $Y = X - X^\star$, that is,

$$\omega^1_{X/D} = \Omega^1_X(\log(Y))/\Omega^1_D(\log(0)).$$

We fix a uniformizer $t : D \to \mathbb{C}$, a universal cover $\bar{D}^\star \to D^\star$ and a logarithm $\log t$ on $\bar{D}^\star$. In [36], Steenbrink constructed an isomorphism $(\omega^{\cdot}_{X/D})_{|Y} \longrightarrow \mathsf{R}\Psi_f \mathbb{C}$ depending on these choices. From this, he deduced a mixed Hodge structure on $\mathrm{H}^q(Y, (\omega^{\cdot}_{X/D})_{|Y})$ which is by definition the limit of the above Variation of Hodge Structures.

1.2.4 The analogy between the situations in étale cohomology and Hodge theory

Let V be a smooth projective variety defined over a field k of characteristic zero. Suppose also given an algebraic closure $\bar{k}/k$ with Galois group G_k and an embedding $\sigma : k \subset \mathbb{C}$. In the étale case, the ℓ-adic cohomology of $V_{\bar{k}}$ is equipped with a structure of a continuous G_k-module. In the complex analytic case, the Betti cohomology of $V(\mathbb{C})$ is equipped with a Hodge structure.

Now let $f : X \longrightarrow C$ be a flat and proper family of smooth varieties over k parametrized by an open k-curve C. Then for any $\bar{k}$-point t of C, we have a continuous Galois module‡ $\mathrm{H}^q(X_t, \mathbb{Q}_\ell)$. These continuous Galois modules can be thought of as a "Variation of Galois Representations" parametrized by C which is the étale analogue of the Variation of Hodge structures $(\mathrm{H}^q(X_t(\mathbb{C}), \mathbb{Q}), F^k)$ that we discussed in the above paragraph.

Now let s be a point of the boundary of C and choose a uniformizer near s. As in the Hodge–theoretic case, the variation of Galois modules above has a "limit" on s which is a "mixed" Galois module given by the following data:

- A monodromy operator N which is nilpotent. This operator induces the monodromy filtration which turns out to be compatible with the weight

‡ In general only an open subgroup of G_k acts on the cohomology, unless t factors trough a k-rational point.

filtration of Steenbrink's mixed Hodge structure on the limit cohomology (see [15]),

- The grading associated to the monodromy filtration is a continuous Galois module of "pure" type.

As in the analytic case, this limit is defined via the nearby cycles complex. Indeed, choose an extension of f to a projective scheme X' over $C' = C \cup \{s\}$. Let Y be the special fiber of X'. The choice of a uniformizer gives us a complex $\mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell$ on Y. Then the "limit" of our "Variation of Galois representations" is given by $H^q(Y, \mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell)$. The monodromy operator N is induced from the representation on $\mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell$ of the étale fundamental group of the punctured henselian neighbourhood of s in C.

1.3 Specialization systems

The goal of this section is to axiomatize some formal properties of the nearby cycles functors that we expect to hold in the motivic context. The result will be the notion of specialization systems. We then state some consequences of these axioms which play an important role in the theory. Before doing that we recall briefly the motivic categories we use.

1.3.1 The motivic categories

Let X be a noetherian scheme. In this paper we will use two triangulated categories associated to X:

(i) The motivic stable homotopy category $\mathbf{SH}(X)$ of Morel and Voevodsky,
(ii) The stable category of mixed motives $\mathbf{DM}(X)$ of Voevodsky.

These categories are respectively obtained by taking the *homotopy category* (in the sense of Quillen [31]) associated to the two *model categories* of $T = (\mathbb{A}^1_X/\mathbb{G}\mathrm{m}_X)$-spectra:

(i) The category $\mathbf{Spect}^T_s(X)$ of T-spectra of simplicial sheaves on the smooth Nisnevich site $(\mathrm{Sm}/X)_{\mathrm{Nis}}$,
(ii) The category $\mathbf{Spect}^T_{\mathrm{tr}}(X)$ of T-spectra of complexes of sheaves with transfers on the smooth Nisnevich site $(\mathrm{Sm}/X)_{\mathrm{Nis}}$.

Recall that a T-spectrum E is a sequence of objects $(E_n)_{n\in\mathbb{N}}$ connected by maps of the form $E_n \longrightarrow \underline{\mathrm{Hom}}(T, E_{n+1})$. We sometimes denote by $\mathbf{Spect}^T(X)$ one of the two categories $\mathbf{Spect}^T_s(X)$ or $\mathbf{Spect}^T_{\mathrm{tr}}(X)$. We do not intend to give the detailed construction of these model categories as this has already been done in several places (cf. [5], [20], [24], [25], [28], [33], [37]). For the reader's convenience, we however give some indications. We focus mainly on the class of weak equivalences; indeed this is enough to define the homotopy category which is obtained by formally inverting the arrows in this class. The weak equivalences in these two categories of T-spectra are called the stable $\mathbb{A}^1$-weak equivalences and are defined in the three steps. We restrict ourself to the case of simplicial sheaves; the case of complexes of sheaves with transfers is completely analogous.

Step 1. We first define simplicial weak equivalences for simplicial sheaves. A map $A_\bullet \longrightarrow B_\bullet$ of simplicial sheaves on $(\mathrm{Sm}/X)_{\mathrm{Nis}}$ is a simplicial weak equivalence if for any smooth X-scheme U and any point $u \in U$, the map of simplicial sets‡ $A_\bullet(\mathrm{Spec}(\mathcal{O}^h_{U,u})) \longrightarrow B_\bullet(\mathrm{Spec}(\mathcal{O}^h_{U,u}))$ is a weak equivalence (i.e. induces isomorphisms on the set of connected components and on the homotopy groups).

Step 2. Next we perform a Bousfield localization of the simplicial model structure on simplicial sheaves in order to invert the projections $\mathbb{A}^1_U \longrightarrow U$ for smooth X-schemes U (see [13] for a general existence theorem on localizations and [28] for this particular case). The model structure thus obtained is the $\mathbb{A}^1$-model structure on simplicial sheaves over $(\mathrm{Sm}/X)_{\mathrm{Nis}}$. We denote $\mathbf{Ho}_{\mathbb{A}^1}(X)$ the associated homotopy category.

Step 3. If A is a pointed simplicial sheaf and $E = (E_n)_n$ is a T-spectrum of simplicial sheaves we define the stable cohomology groups of A with values in E to be the colimit: $\mathsf{Colim}_n \hom_{\mathbf{Ho}_{\mathbb{A}^1}(X)}(T^{\wedge n} \wedge A, E_n)$. We then say that a morphism of spectra $(E_n)_n \longrightarrow (E'_n)_n$ is a stable $\mathbb{A}^1$-weak equivalence if it induces isomorphisms on cohomology groups for every simplicial sheaf A.

By inverting stable $\mathbb{A}^1$-weak equivalences in $\mathbf{Spect}^T_s(X)$ and $\mathbf{Spect}^T_{\mathrm{tr}}(X)$ we get respectively the categories $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$. Let U be a smooth X-scheme. We can associate to U the pointed simplicial sheaf U_+ which is simplicially constant, represented by $U \coprod X$ and pointed by the trivial map $X \longrightarrow U \coprod X$. Then, we can associate to U_+ its infinite T-suspension $\Sigma^\infty_T(U_+)$ given in level n by $T^{\wedge n} \wedge U_+$. This provides a covariant functor $M : \mathrm{Sm}/X \longrightarrow \mathbf{SH}(X)$ which associates to U its motive $M(U)$. Similarly we can associate to U the complex $\mathbb{Z}_{\mathrm{tr}}(U)$, concentrated in degree zero, and then take its infinite suspension given in level n by $\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^n \times U)/\mathbb{Z}_{\mathrm{tr}}((\mathbb{A}^n - 0) \times$

‡ This map of simplicial sets is the stalk of $A_\bullet \longrightarrow B_\bullet$ at the point $u \in U$ with respect to the Nisnevich topology.