Problem solving and mathematical thinking

The purpose of this *Maths Trail* is to help give some meaning to problem solving with a particular focus on visualising. This book belongs in a series of titles that looks at particular skills needed in the problem-solving process. Other publications in the series include *Generalising* and *Working Systematically*. While this series tries to give detail to particular mathematical thinking skills, working through each book will not on its own make an expert problem solver.

We want pupils to engage in doing mathematics in a way that means more than the regurgitation of facts or the use of certain skills in particular contexts (for example, finding the area of a rectangle by remembering to multiply length by width). This book is about encouraging pupils to 'be mathematical' in the sense that they:

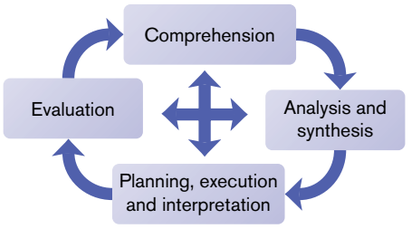
- think about and communicate ideas;
- engage in problem-solving activities;
- create and identify mathematical problems within given contexts.

Clearly, being mathematical and doing mathematics in this sense involves some mathematical content knowledge. But if mathematics is more than learning facts and practising skills, how can we support learners in 'being mathematical'? We need to offer them the opportunity to be problem solvers who can pose and explore problems. If we are to support problem solving effectively we need some guidance and structure upon which we can focus.

What is problem solving?

When we are presented with a mathematical problem, it is only a problem if we do not immediately know how to solve it. This means that something that is a problem to one person may not be a problem to another. The process of problem solving is like a journey from a state of not knowing what to do towards a destination which we hope will be a solution. The key is to have some strategies at our fingertips that will help us to get started and to identify a possible route to a solution. Our mathematical journey is often full of twists and turns, where we revisit ideas or need to step back and look for alternatives. Often a mistake or dead-end gives vital clues to the mathematics of the problem and is therefore crucial in any solution process.

To help us have a sense of direction when we are problem solving and decide what might be a good strategy to try next, it can be useful to have a model. There are many such models but the one below is a good starting point. Although the descriptions on the next page are written in a linear sequence, the diagram emphasises the reality that problem solving can be a messy process which often involves revisiting places on the journey from problem posing to problem solving.



Comprehension

This stage is about making sense of the problem by using strategies such as retelling, identifying relevant information and creating mental images. Pupils can be helped at this stage by being encouraged to re-read the problem several times and to record in some way what they understand the problem to be about (for example, by drawing a picture or making notes).

Analysis and synthesis

This stage is about 'homing in' on what the problem is asking solvers to investigate.

- Can they represent the situation mathematically?
- What is it that they are trying to find?
- What do they think the answer might be (conjecturing and hypothesising)?
- What might they need to find out before they can get started?

Central to this stage is identifying what is unknown and what needs finding.

Planning, execution and interpretation

Once pupils have understood what the problem is about and have established what needs finding, the next stage is planning and executing a pathway to the solution. It is within this process that you might encourage pupils to think about whether they have seen something similar before and what strategies they adopted then. This will help them to identify appropriate methods and tools. Particular knowledge and skills gaps that need addressing may become evident at this stage.

During the execution phase, pupils might identify further related problems they wish to investigate. They will need to consider how they will keep track of what they have done and how they will communicate their findings. This phase of the problem-solving process ends with interpreting results and drawing conclusions.

Evaluation

Pupils can learn as much from reflecting on and evaluating what they have done as they can from the process of solving the problem itself. During this phase pupils should be expected to reflect on the effectiveness of their own approach as well as that of other people, to justify their conclusions and to assess their own learning. Evaluation may also lead to thinking about other questions that could now be investigated.

Using the model

The use of this model has a number of benefits in the classroom:

- It is a structure that can help pupils frame their problem solving and keep track of where they are in the process.
- It gives us a language which helps us to talk to pupils about what they are doing. For example:
  - 'Can you tell me what you think the problem is about?'
  - 'What are you trying to find?'
  - 'Have you seen anything like this before and what did you do then?'
  - 'Could you have solved this in a different way?'
- It offers a framework which can help structure lessons so that you can plan what pupils might do at various stages.

To support you in identifying the phases in the model, the lesson outlines indicate some key aspects of problem solving. We think it is worth drawing the attention of pupils to the model during their work on each task. Many of the activities involve engagement with all aspects of the problem-solving 'cycle'. The aspects highlighted are intended to offer a particular focus for a lesson.

What is mathematical thinking?

Within the problem-solving framework, there are many mathematical skills which pupils need to have at their fingertips. These skills involve more than numeric, geometric and algebraic manipulation. They include strategies such as:

- modelling;
- visualising;
- working systematically;
- generalising.

We would class these skills as elements of mathematical thinking that are needed to engage in the problem-solving process. This book focuses on the skills associated with *visualising*.

What is visualising?

We use some visualisation strategies almost every time we engage in problem solving. Visualisation can have a variety of purposes and involves having access to a range of skills.

Purposes of visualising

1 Modelling a situation
This is particularly useful when the situation is physically unattainable. In other words we try to see the 'unseeable', for example the inside of a 3-D object, or a very large number.

2 Stepping into a problem
Using visualisation aids understanding at the comprehension stage. Here visualisation gives us space to go more deeply into the situation, to clarify and support understanding.

3 Planning ahead
Visualising can be used throughout a problem to anticipate and to plan ahead by asking 'What will be the consequence of doing this?'.

Visualising skills

The examples in this book emphasise the purposes of visualising described above. However, they also aim to encourage the development of specific skills that underpin these purposes. The list below describes in more detail these useful visualising skills:

A Focusing on a problem or idea without talking for several minutes.

B Identifying a useful image or representation of an idea. The image might be suggested by someone else but it still has some meaning for you. This may be a representation that helps you see the larger picture and/or describe the structure of the problem.

C Scrutinising different images to identify what is the same or different. This includes:
C1 Comparing other people's representations with our own.

C2 Identifying the general and the specific in a representation and what is significant in terms of the problem at hand. For example, recognising that when two squares are put together, although there is an infinite number of possible orientations (general) you only need to consider one and to have that clearly in your mind (specific).

C3 Trying to hold more than one image in your head. For example, remembering a starting point and being able to get back again when your ideas don't work, or seeing and/or reproducing a logical sequence.

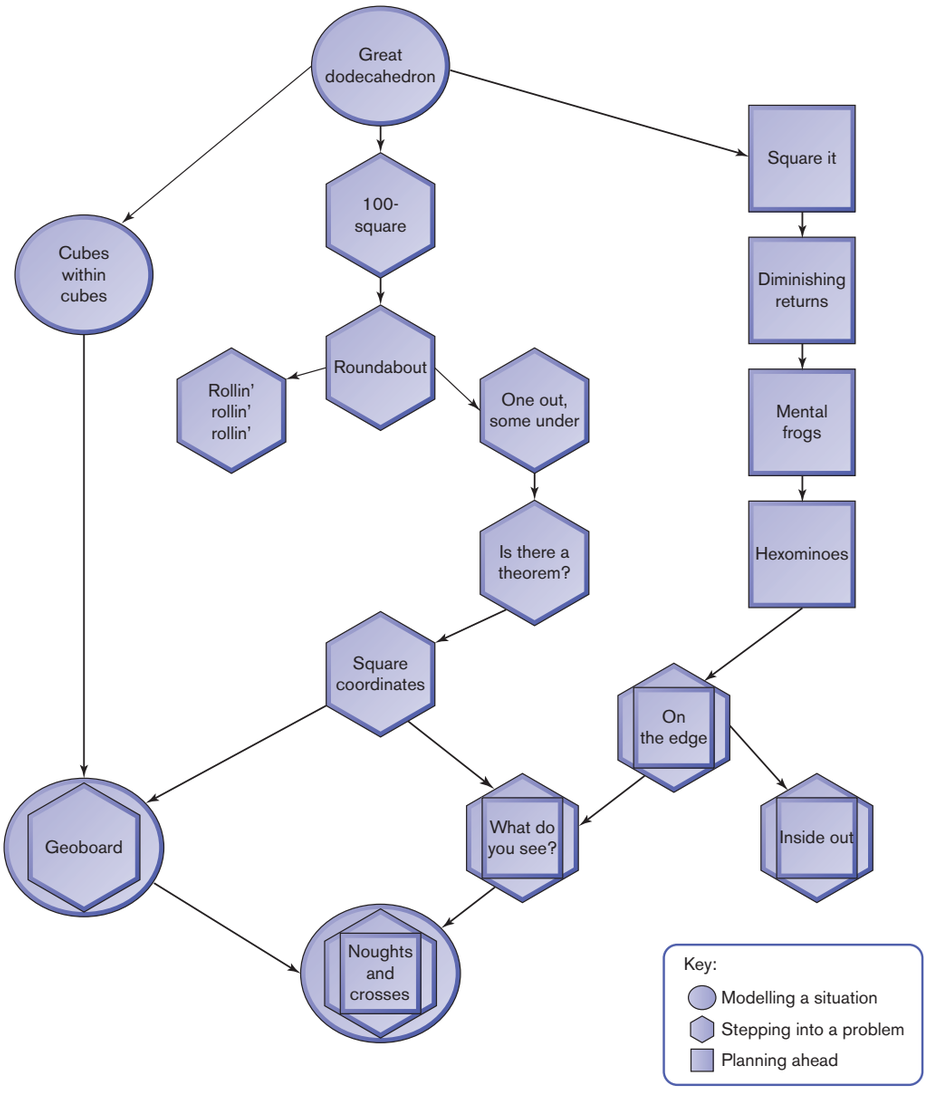
D Making connections. For example, remembering the processes or underpinning structures rather than individual images.

E Describing a representation (your own or someone else's) to an audience. For example, to explain or clarify thinking or share interpretations.

The visualising trail

The trail structure is shown below and runs from top to bottom. It includes problems focusing on the three purposes of visualising and covers the range of visualising skills described earlier.

In general, a problem which is further down the trail and coded with the same shape requires higher efficiency in that aspect of visualising or involves some aspect of mathematics associated with a higher level of maturity.



Why use this trail?

We often present a problem to pupils and then jump straight into thinking about how to solve it without giving them time to create a picture of what is going on in their own minds. One thing the lessons in this book encourage is giving pupils that time and some strategies for 'getting to know' a problem by using their visualising skills. It can be frustrating when pupils jump into problems without thinking ahead and considering 'what would happen if' because as a result they often get nowhere quickly. Visualising is a strategy we need to encourage pupils to employ throughout problem solving. Mental images can also act as a focus for discussion and sharing.

The table below lists the problems in the trail and gives suggestions for the major visualisation foci of each one. The numbers and letters refer to the purposes of visualising and the visualising skills listed in the section 'What is visualising?'. Clearly a slightly different emphasis during teaching can significantly change the visualisation focus.

	Purposes			Skills						
	1	2	3	A	B	C1	C2	C3	D	E
Great dodecahedron	X									X
Cubes within cubes	X			X		X			X	X
100-square		X		X			X		X	
Roundabout		X		X		X		X		X
Rollin' rollin' rollin'		X		X		X		X		X
One out, some under		X		X					X	
Is there a theorem?		X				X		X		X
Square coordinates		X						X		
Square it			X				X			
Diminishing returns			X						X	X
Mental frogs			X		X			X		
Hexominoes			X	X				X	X	X
On the edge		X	X				X		X	
Inside out		X	X				X		X	
What do you see?		X	X	X			X	X	X	X
Geoboard		X	X		X	X	X			X
Noughts and crosses	X	X	X							X