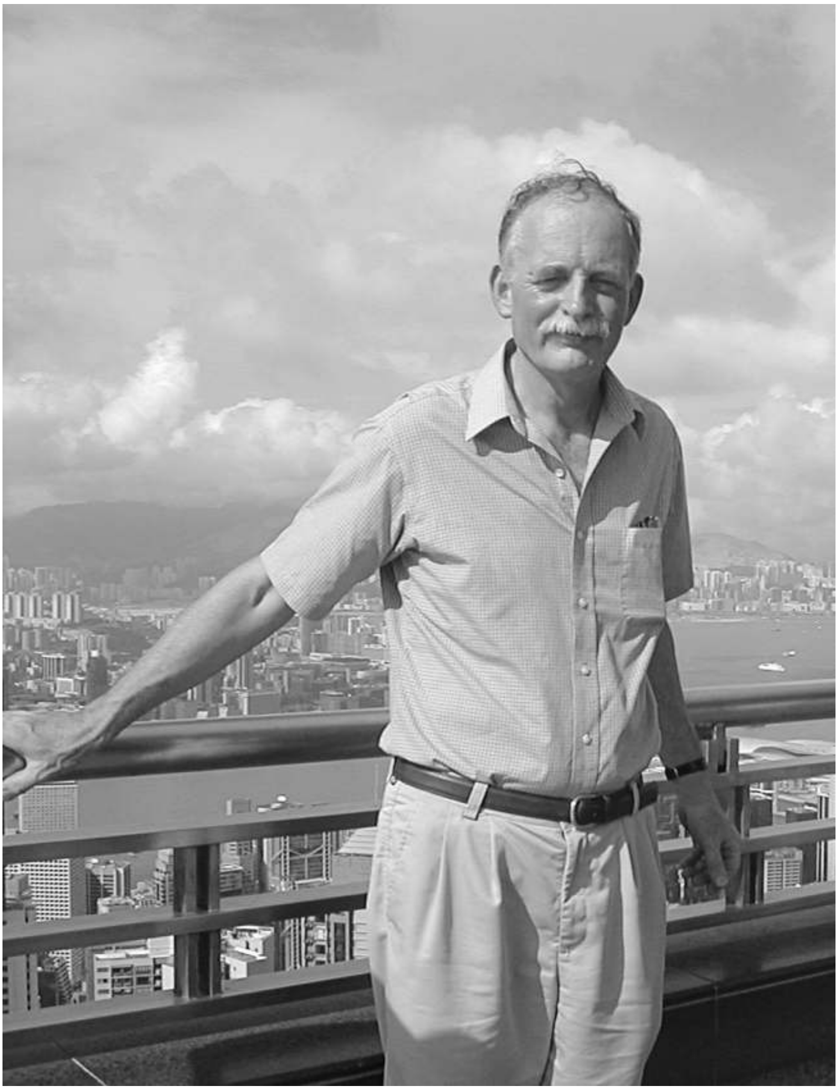

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DISCRETE TORSION FOR THE SUPERSINGULAR ORBIFOLD SIGMA GENUS

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Abstract. The first purpose of this paper is to examine the relationship between equivariant elliptic genera and orbifold elliptic genera. We apply the character theory of [HKR00] to the Borel-equivariant genus associated to the sigma orientation of [AHS01] to define an orbifold genus for certain total quotient orbifolds and supersingular elliptic curves. We show that our orbifold genus is given by the same sort of formula as the orbifold "two-variable" genus of [DMVV97] and [BL02]. In the case of a finite cyclic orbifold group, we use the characteristic series for the two-variable genus in the formulae of [And03] to define an analytic equivariant genus in Grojnowski's equivariant elliptic cohomology, and we show that this gives *precisely* the orbifold two-variable genus. The second purpose of this paper is to study the effect of varying the $BU\langle 6\rangle$-structure in the Borel-equivariant sigma orientation. We show that varying the $BU\langle 6\rangle$ structure by a class in $H^3(BG;\mathbb{Z})$, where G is the orbifold group, produces discrete torsion in the sense of [Vaf85]. This result was first obtained by Sharpe [Sha], for a different orbifold genus and using different methods.

1. Introduction

Let E be an even periodic, homotopy commutative ring spectrum, let C be an elliptic curve over $S_E = \operatorname{spec} \pi_0 E$, and let t be an isomorphism of formal groups

$$t : \widehat{C} \cong \operatorname{spf} E^0(\mathbb{C}P^\infty),$$

so that $\mathbf{C} = (E, C, t)$ is an elliptic spectrum in the sense of [Hop95, AHS01]. In [AHS01], Hopkins, Strickland, and the first author construct a map of homotopy commutative ring spectra

$$\sigma(\mathbf{C}) : MU\langle 6\rangle \to E$$

called the *sigma orientation*; it is conjectured in [Hop95] that this map is the restriction to $MU\langle 6\rangle$ of a similar map $MO\langle 8\rangle \to E$. ($MU\langle 6\rangle$ is the bordism

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theory of SU-manifolds with a trivialization of the second Chern class, and $MO\langle 8\rangle$ is the bordism theory of spin manifolds with a trivialization of the characteristic class $\frac{p_1}{2}$.)

The sigma orientation is natural in the elliptic spectrum, and, if $K_{\mathrm{Tate}} = (K[\![q]\!], \mathrm{Tate}, t)$ is the elliptic spectrum associated to the Tate elliptic curve, then the map of homotopy rings

$$\pi_* MU\langle 6\rangle \to \pi_* K_{\mathrm{Tate}} \tag{1.1}$$

is the restriction from $\pi_* MSpin$ of the *Witten genus*. Explicitly, let M be a Riemannian spin manifold, and let D be its Dirac operator. Let T denote the tangent bundle of M. If V is a (real or complex) vector bundle over M, let $V^{\mathbb{C}}$ be the complex vector bundle

$$V^{\mathbb{C}} = V \underset{\mathbb{R}}{\otimes} \mathbb{C}.$$

If V is a complex vector bundle, let rV be the reduced bundle

$$rV = V - \underline{\operatorname{rank} V},$$

and let

$$S_t V = \sum_{k\geq 0} t^k S^k V$$

be the indicated formal power series in the symmetric powers of V. The operation $V \mapsto S_t V$ extends to an exponential operation

$$K(X) \to K(X)[\![t]\!]$$

because of the formula

$$S_t(V \oplus W) = (S_t V)(S_t W).$$

The *Witten genus* of M is given by the formula

$$w(M) = \operatorname{ind}\left(D \otimes \bigotimes_{k\geq 1} S_{q^k}(rT^{\mathbb{C}})\right) \in \mathbb{Z}[\![q]\!], \tag{1.2}$$

and the diagram

$$\begin{array}{ccc} \pi_* MU\langle 6\rangle & \longrightarrow & \pi_* MSpin \\ & \searrow^{\pi_*\sigma(K_{\mathrm{Tate}})} & \downarrow w \\ & & \mathbb{Z}[\![q]\!] \end{array}$$

commutes [AHS01].

The Witten genus first arose in [Wit87], where Witten showed that various elliptic genera of a manifold M are essentially one-loop amplitudes of quantum field theories of closed strings moving in M. Locally on a spin manifold M, the quantum field theory associated to $w(M)$ is a conformal field theory, and the obstruction to assembling a conformal field theory globally on M is $c_2 M$.

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[1] This gives a physical proof that, if $c_2 M = 0$, then $w(M)$ is the q-expansion of a modular form.

Suppose that M is an SU-manifold. The formula (1.2) shows that the Witten genus is an invariant of the spin structure of M. On the other hand the sigma orientation depends on a choice of $BU\langle 6\rangle$ structure, that is, a lift in the diagram

$$\begin{array}{ccccc} & & BU\langle 6\rangle & & \\ & \nearrow & \downarrow & & \\ M & \longrightarrow & BSU & \xrightarrow{c_2} & K(\mathbb{Z},4). \end{array}$$

It is an interesting problem to understand how the orientation depends on this choice. The fibration sequence

$$K(\mathbb{Z},3) \xrightarrow{\iota} BU\langle 6\rangle \to BSU \xrightarrow{c_2} K(\mathbb{Z},4)$$

shows that a lift exists precisely when $c_2(M) = 0$, and that the set of lifts is a quotient of $H^3(M;\mathbb{Z})$.

The dependence on the choice of lift appears to have an explanation in string theory. The action for the QFT described by the Witten genus is a function on the space of maps

$$X : \Sigma \to M$$

of 2-dimensional surfaces Σ to M. If the theory is anomaly-free, that is, if $c_2 M = 0$, then one is free to add to the action a term of the form

$$\int_\Sigma X^* B,$$

where $B \in \Omega^2 M$ is a differential 2-form on M (called the "B-field"), provided that

$$H = dB$$

is an *integral* three-form. It seems clear that the physics of the B-field should account for the variation in the sigma orientation from at least torsion classes in $H^3(M,\mathbb{Z})$.

This situation has already received a good deal of attention, particularly in the case of orbifolds. Eric Sharpe [Sha] showed that Vafa's *discrete torsion* [Vaf85] arises from, as he put it us once, "the action of the orbifold group on the B-field." Lupercio and Uribe [LU] have explained how discrete torsion arises from a gerbe on a total quotient orbifold M/G with finite orbifold group (and so gives rise to a class in the Borel cohomology group $H^3(M_G;\mathbb{Z})$).

The techniques used to date in the study of the sigma orientation have quite a different flavor from those used in the study of gerbes on orbifolds.

[1] There are various ways to understand this obstruction ([Wit87, BM94, GMS00, And03]).

One goal of the research which led to this paper was to find using the sigma orientation some of the phenomena associated to the B-field. In this direction, we show that varying the $BU\langle 6\rangle$ structure in the *orbifold* sigma genus of a supersingular elliptic curve indeed produces a variant of discrete torsion.

More precisely, suppose that M is a complex manifold with an action by a group G, and suppose that V is a complex G-vector bundle over M. Let T denote the tangent bundle of M. If X is a space, let X_G denote the Borel construction $EG \times_G X$. If

$$c_1(T_G) = c_1(V_G)$$
$$c_2(T_G) = c_2(V_G),$$

then there is a lift in the diagram

$$\begin{array}{ccc} & & BU\langle 6\rangle \\ & \overset{\ell}{\nearrow} & \downarrow \\ M_G & \xrightarrow[V_G - T_G]{} & BSU, \end{array} \tag{1.3}$$

and a choice of lift gives a Thom class

$$U(M, \ell, \mathbf{C})_G \in E^0(V_G - T_G).$$

The relative zero section together with the Pontrjagin-Thom construction provide a map

$$\tau(V)_G : E^0(V_G - T_G) \to E^0(-T_G) \to E^0(BG),$$

and

$$\tau(V)_G(U(M, \ell, \mathbf{C})_G) \in E^0(BG)$$

is the (Borel) equivariant sigma genus of M twisted by V (see §4).

To get from it an "orbifold" genus taking its values in E^0, we use the character theory of Hopkins, Kuhn, and Ravenel ([HKR00]; see also §5). It associates to a pair (g, h) of commuting elements of G a ring homomorphism

$$\Xi_{g,h} : E^0(BG) \to D,$$

where D is a complete local E-algebra which depends on the formal group of the spectrum E. It turns out that the quantity

$$\sigma_{\mathrm{orb}}(M, \ell, \mathbf{C})_G = \sum_{gh=hg} \Xi_{g,h}\tau(V)_G(U(M, \ell, \mathbf{C})_G) \tag{1.4}$$

takes its values in E^0; we call it the *orbifold sigma genus* of M twisted by V (see §6).

There is already an extensive literature on the subject of "orbifold elliptic genera", particularly the "two-variable" elliptic genus of [Kri90, EOTY89]; see for example [DMVV97, BL02]. In §6, we show that the formula (1.4) is formally analogous to the formula for the orbifold two-variable genus. More precisely, by applying the topological Riemann-Roch formula to the expression (1.4) and using the Thom class associated to the the two-variable elliptic genus in place of U, we obtain a formula nearly identical to that of [BL02].

Unfortunately our use of the topological Riemann-Roch formula in this context, while illuminating, is not useful for calculation, because we work with the Borel-equivariant elliptic cohomology associated to a supersingular elliptic curve, which is a highly completed situation. In order to locate the orbifold two-variable genus more precisely in the setting of equivariant elliptic cohomology, we consider in §7 the case of a finite cyclic group $G = \mathbb{T}[n] \subset \mathbb{T}$. We use the principle suggested by Shapiro's Lemma to define

$$E_G(X) \stackrel{\text{def}}{=} E_{\mathbb{T}}(\mathbb{T} \times_G X),$$

where $E_{\mathbb{T}}$ is the uncompleted analytic equivariant elliptic cohomology of Grojnowski. We adapt the formulae in [And03], which descends from [Ros01, AB02], to write down an euler class in $E_G(X)$. The associated genus $\mathrm{Ell}^{an}(M, G)$ takes its value in $\Gamma(E_G(*)) \cong \Gamma(\mathcal{O}_{C[n]}) \cong \mathbb{C}^{G \times G}$, and we prove the following.

Theorem 1.5. *Suppose that G is a finite cyclic group. Summing the analytic equivariant two-variable genus over the torsion points of the elliptic curve gives the orbifold two-variable genus: more precisely, we have*

$$\mathrm{Ell}_{orb}(X, G) = \frac{1}{|G|} \sum_{gh=hg} \mathrm{Ell}^{an}(M, G, g, h).$$

We were pleased to be able to confirm that orbifold elliptic genera are (in the cyclic case) so simply obtained from equivariant elliptic genera. It would be interesting to use this observation to investigate more subtle properties of orbifold genera, such as, for example, the "McKay correspondence" of Borisov and Libgober.

The rest of the paper is devoted to the study of the dependence of the orbifold sigma genus on the choice ℓ of $BU\langle 6\rangle$ structure in (1.3). Suppose that we have chosen an element $u \in H^3(BG; \mathbb{Z})$, represented as a map

$$u : BG \to K(\mathbb{Z}, 3).$$

If $\pi : M_G \to BG$ denotes the projection in the Borel construction, then we obtain an element

$$\pi^* u = u\pi \in H^3(M_G; \mathbb{Z}),$$

and $\ell + \iota u\pi$ is another $BU\langle 6\rangle$-structure on $V_G - T_G$.

In §8, we use the character theory and the sigma orientation to associate to u an alternating bilinear map

$$\delta = \delta(u, \mathbf{C}) : G_2 \to D^\times,$$

where G_2 denotes the set of pairs of commuting elements of G of p-power order. In §9 we obtain the

Theorem 1.6. *The orbifold sigma genus associated to the $BU\langle 6\rangle$ structure $\ell + \iota u\pi$ is related to the equivariant sigma genus associated to ℓ by the formula*

$$\sigma_{\mathrm{orb}}(M, \ell + \iota u\pi, \mathbf{C})_G = \sum_{gh=hg} \delta(g,h)\Xi_{g,h}\tau(V)_G(U(M,\ell,\mathbf{C})_G).$$

In [Vaf85], Vafa observed that if

$$\phi = \sum_{gh=hg} \phi_{g,h}$$

is an orbifold elliptic genus associated to a theory of strings on M, and if $c = c(g,h)$ is a 2-cocycle with values in $U(1)$, then

$$\sum_{gh=hg} c(g,h)\phi_{g,h} \tag{1.7}$$

is again modular; he called this phenomenon "discrete torsion". Eric Sharpe [Sha] showed that the genus (1.7) arises from adding a B-field on the orbifold M/G. As expressed by Lupercio and Uribe, this is a $U(1)$-gerbe B on the orbifold M/G, whose associated cohomology class $[B] \in H^3(M_G;\mathbb{Z})$ satisfies

$$[B] = [c] \in H^3(M_G;\mathbb{Z}),$$

where $[c]$ is the cohomology class in M_G obtained from c by pulling back along $M_G \to BG$.

Our result shows that varying the $BU\langle 6\rangle$-structure of M_G by an element $u \in H^3(BG)$ has a similar effect on the orbifold sigma genus. When G is an abelian group of order dividing $n = p^s$, the map δ may be viewed as a two-cocycle on G with values in $D^\times[n] \cong \mathbb{Z}/n$, and as such it represents a cohomology class in $H^2(BG;\mathbb{Z}/n) \cong H^3(BG)$. It is not quite the cohomology class u: instead, as we shall see in §10, if c is a 2-cocycle representing $u \in H^3(BG;\mathbb{Z}) \cong H^2(BG;\mathbb{Z}/n)$, then

$$\delta(g,h) = c(g,h) - c(h,g).$$

2. The sigma orientation and the sigma genus

In this section we recall some results from [AHS01].

Definition 2.1. An *elliptic spectrum* consists of

(1) an even, periodic, homotopy commutative ring spectrum E with formal group $P_E = \operatorname{spf} E^0\mathbb{C}P^\infty$ over $\pi_0 E$;
(2) a generalized elliptic curve C over $\pi_0 E$;
(3) an isomorphism $t : P_E \to \widehat{C}$ of P_E with the formal completion of C.

A *map* (f, s) of elliptic spectra $\mathbf{E_1} = (E_1, C_1, t_1) \to \mathbf{E_2} = (E_2, C_2, t_2)$ consists of a map $f : E_1 \to E_2$ of multiplicative cohomology theories, together with an isomorphism of elliptic curves

$$C_2 \xrightarrow{s} (\pi_0 f)_* C_1,$$

extending the induced isomorphism of formal groups.

Theorem 2.2. *An elliptic spectrum* $\mathbf{C} = (E, C, t)$ *determines a map*

$$\sigma(\mathbf{C}) : MU\langle 6\rangle \to E$$

of homotopy-commutative ring spectra. The association $\mathbf{C} \mapsto \sigma(\mathbf{C})$ *is modular, in the sense that if*

$$(f, s) : \mathbf{C_1} \to \mathbf{C_2}$$

is a map of elliptic spectra, then the diagram

$$\begin{array}{ccc} MU\langle 6\rangle & \xrightarrow{\sigma(\mathbf{C_1})} & E_1 \\ & \searrow^{\sigma(\mathbf{C_2})} & \downarrow f \\ & & E_2 \end{array}$$

commutes up to homotopy. If $K_{\mathrm{Tate}} = (K[\![q]\!], \mathrm{Tate}, t)$ *is the elliptic spectrum associated to the Tate curve, then the diagram*

$$\begin{array}{ccc} MU\langle 6\rangle & \longrightarrow & MSpin \\ & \searrow^{\sigma(K_{\mathrm{Tate}})} & \downarrow w \\ & & K[\![q]\!] \end{array}$$

commutes, where w *is the orientation associated to the Witten genus.*

3. The sigma genus

Definition 3.1. Let W be a virtual complex vector bundle on a space M. A $BU\langle 6\rangle$*-structure* on W is a map

$$\ell : M \to BU\langle 6\rangle$$

such that the composition

$$M \xrightarrow{\ell} BU\langle 6\rangle \to BU$$

classifies rW.

Now let M be a connected compact closed manifold with complex tangent bundle T, and let V be another complex vector bundle on M. Let

$$d = 2\,\mathrm{rank}_{\mathbb{C}}\, T - V$$

Let

$$\tau(V) : S^0 \xrightarrow{P-T} M^{-T} \xrightarrow{\zeta} M^{V-T}$$

be the composition of the Pontrjagin-Thom map with the relative zero section.

If
$$\ell : M \to BU\langle 6\rangle$$
is a $BU\langle 6\rangle$-structure on $V - T$, and if $\mathbf{C} = (E, C, t)$ is an elliptic spectrum, let $U(M, \ell, \mathbf{C}) \in E^{-d}(M^{V-T})$ be the class given by the map
$$U(M, \ell, \mathbf{C}) : \Sigma^d M^{V-T} \xrightarrow{\ell} MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{C})} E.$$

Definition 3.2. The *sigma genus* of ℓ in $\mathbf{C}$ is the element
$$\sigma(M, \ell, \mathbf{C}) \stackrel{\text{def}}{=} \tau(V)^*(U(M, \ell, \mathbf{C})) \in E^{-d}(S^0) = \pi_d E.$$

Example 3.3. Suppose that $c_1 T = 0 = c_2 T$, so that T itself admits a $BU\langle 6\rangle$-structure, say $\ell : M \to BU\langle 6\rangle$, and $d = 2 \dim M$. Then we have a Thom isomorphism
$$\begin{aligned} E^0(M) &\cong E^{-d}(M^{-T}) \\ 1 &\mapsto U(M, \ell, \mathbf{C}). \end{aligned}$$
and the usual Umkehr map $\pi_!^M$ associated to the projection
$$\pi^M : M \to *$$
is the composition
$$\pi_!^M : E^0(M) \xrightarrow{\cong} E^{-d}(M^{-T}) \xrightarrow{P-T} E^{-d}(S^0) \cong \pi_d E.$$
Thus
$$\sigma(M, \ell, \mathbf{C}) = \pi_!^M(1) = \pi_d(\sigma(\mathbf{C}))([M]) \in \pi_d E$$
is just the genus of M with $BU\langle 6\rangle$-structure ℓ, associated to the sigma orientation
$$MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{C})} E.$$

4. The Borel-equivariant sigma genus

Now suppose that G is a compact Lie group, and, if X is a space, let X_G denote the Borel construction
$$X_G \stackrel{\text{def}}{=} EG \times_G X.$$
Suppose that G acts on the compact connected manifold M, that V is an equivariant complex vector bundle, and that
$$\ell : M_G \to BU\langle 6\rangle$$
is a $BU\langle 6\rangle$-structure on the bundle $V_G - T_G$. Since T_G is the bundle of tangents along the fiber of
$$M_G \to BG,$$
we have a Pontrjagin-Thom map
$$BG_+ \xrightarrow{P-T} (M_G)^{-T_G},$$

and so a map

$$\tau(V)_G : BG_+ \xrightarrow{P-T} (M_G)^{-T_G} \xrightarrow{\zeta} (M_G)^{V_G - T_G}.$$

Let $U(M,\ell,\mathbf{C})_G \in E^{-d}(M_G^{V_G-T_G})$ be given by the map

$$U(M,\ell,\mathbf{C})_G : \Sigma^d (M_G)^{V_G-T_G} \xrightarrow{\ell} MU\langle 6\rangle \xrightarrow{\sigma(\mathbf{C})} E.$$

Definition 4.1. The *(Borel) equivariant sigma genus* of ℓ in $\mathbf{C}$ is the element

$$\sigma(M,\ell,\mathbf{C})_G \overset{\text{def}}{=} \tau(V)_G(U(M,\ell,\mathbf{C})_G) \in E^{-d}(BG).$$

5. Character theory

The equivariant sigma genus described in §4 is not so familiar, because $E^*(BG)$ is not. In this section we review the character theory of [HKR00], which gives a sensible way to understand $E^*(BG)$. In the next section, we apply the character theory to produce the orbifold sigma genus from the equivariant sigma genus; as we shall see, it is given by the same sort of formula as those for "orbifold elliptic genera" in for example [DMVV97, BL02]

We suppose that E is an even periodic ring spectrum, and that $\pi_0 E$ is a complete local ring of residue characteristic $p > 0$. We write P for $\mathbb{C}P^\infty$, so $P_E = \operatorname{spf} E^0 P$ is the formal group of E. We assume that P_E has finite height h.

Let $\Lambda_\infty = (\mathbb{Z}_p)^h$, and for $n \geq 1$, let $\Lambda_n = \Lambda_\infty / p^n \Lambda_\infty$. If A is an abelian group, let $A^* \overset{\text{def}}{=} \hom(A, \mathbb{C}^\times)$ denote its group of complex characters, so for example $\Lambda_\infty^* \cong (\mathbb{Q}_p/\mathbb{Z}_p)^h \cong (\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})^h$.

Each $\lambda \in \Lambda_n^*$ defines a map

$$B\Lambda_n \xrightarrow{B\lambda} P.$$

Choose a coordinate $x \in E^0 P$. For each $\lambda \in \Lambda_n^*$, let

$$x(\lambda) = (B\lambda)^* x \in E^0 B\Lambda_n.$$

Let $S \subset E^0 B\Lambda_n$ be the multiplicative subset generated by $\{x(\lambda) | \lambda \neq 0\}$. Let $L_n = S^{-1} E^0 B\Lambda_n$, and let D_n be the image of $E^0 B\Lambda_n$ in L_n. In other words, D_n is the quotient of $E^0 B\Lambda_n$ by the ideal generated by annihilators of euler classes of non-zero characters of Λ_n. It is clear that L_n and D_n are independent of the choice of coordinate x.

Now suppose that G is a finite group. Let

$$\alpha : \Lambda_n \to G$$

be a homomorphism: specifying such α is equivalent to specifying an h-tuple of commuting elements of G of order dividing p^n.