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PART I

Basic concepts

CHAPTER 1

Complete contingent claims

In competitive asset markets, consumers make intertemporal choices in an uncertain environment. Their attitudes toward risk, production opportunities, and the nature of trades that they can enter into determine equilibrium quantities and the prices of assets that are traded. The intertemporal choice problem of a consumer in an uncertain environment yields restrictions for the behavior of individual consumption over time as well as determining the form of the asset-pricing function used to price random payoffs.

We begin by describing the simplest setup in which consumer choices are made and asset prices are determined, namely, a complete contingent claims equilibrium for a pure endowment economy. In such an equilibrium, a consumer can trade claims to contracts with payoffs that depend on the state of the world, for all possible states. As a precursor of the material to follow, we discuss the relationship of the complete contingent claims equilibrium to security market equilibrium and describe its implications for asset pricing.

The complete contingent claims equilibrium can also be used to derive restrictions for the behavior of consumption allocations. In this context, we discuss the relationship between the contingent claims equilibrium and Pareto optimality, and show the existence of a "representative consumer" that can be constructed by exploiting the Pareto optimality of the contingent claims equilibrium. Some conclusions follow.

1.1. A ONE-PERIOD MODEL

We initially consider economies with one date and a finite number of states. To understand the nature of the trades that take place in a complete contingent claims equilibrium, imagine that all agents get together at time 0 to write contracts that pay off contingent on some state occurring next period. The realization of the states is not known at the time the contracts are written, although agents know the probabilities and the set of all possible states. Once the contracts are signed, the realization of the state is observed by all agents, and the relevant state-dependent trade is carried out.

We assume the following setup:

- There is a set of I consumers, $\{1, 2, \ldots, I\}$
- Each consumer associates the probability π_s^i to state s occurring, where $0 < \pi_s^i < 1$ and

$$\sum_{s=1}^{S} \pi_s^i = 1.$$

- There are M commodities.
- The notation $c_{s,m}^i$ denotes the consumption of agent i in state s of commodity m.
- A consumption vector for agent i is

$$c^i \equiv \{c_{1,1}^i, \ldots, c_{S,1}^i, c_{1,2}^i, \ldots, c_{S,2}^i, \ldots c_{1,M}^i, \ldots, c_{S,M}^i\},$$

which is a vector of length $S \times M$. Consumption is always non-negative and real so that $c_{s,m}^i \in \Re_+$. The *commodity space* is $\Re_+^{SM}$. The commodity space is the space over which consumption choices are made. When there is a finite number of states (or dates) and a finite number of commodities at each state (or date), we say that the commodity space is finite-dimensional.

- The endowment of agent i is a vector of length $S \times M$,

$$\omega^i = \{\omega_{1,1}^i, \ldots, \omega_{S,M}\}$$

The utility of consumer i is a function $u_i : \Re_+^{SM} \to \Re$,

$$u_i(c^i) = \sum_{s=1}^{S} \pi_s^i U_i(c_{s,1}^i, \ldots, c_{s,M}^i) \tag{1.1}$$

Notice that we assume that utility is additive across states, which is the expected utility assumption.

Here are some definitions.

- An allocation is a vector $(c^1, \ldots, c^I)$.
- An allocation is *feasible* if

$$\sum_{i=1}^{I} [c_{s,m}^i - \omega_{s,m}^i] \leq 0 \tag{1.2}$$

for $s = 1, \ldots, S$ and $m = 1, \ldots, M$. This holds for each commodity and for each state.

- An allocation $(c^1, \ldots, c^I)$ is *Pareto optimal* if there is no other feasible allocation $(\hat{c}^1, \ldots, \hat{c}^I)$ such that

$$u_i(\hat{c}^i) \geq u_i(c^i) \quad \text{for all } i \tag{1.3}$$

and

$$u_i(\hat{c}^i) > u_i(c^i) \quad \text{for some } i. \tag{1.4}$$

1.1.1. Contingent claims equilibrium

Imagine now that agents trade contingent claims – which are agreements of the form that, if state s occurs, agent i will transfer a certain amount of his endowment of good m to agent j. Since there are S states and M commodities in each state, a total of $S \times M$ contingent claims will be traded in this economy. For each state and commodity, let $p_{s,m}$ denote the price of a claim to a unit of consumption of the mth commodity to be delivered contingent on the s'th state occurring. The set of prices $p \in \Re_+^{SM}$ is a *price system*. The price function p assigns a cost to any consumption c^i and a value to any endowment ω^i; in our application $p : \Re_+^{SM} \rightarrow \Re_+$ has an inner product representation:

$$p \cdot c \equiv \sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m} c_{s,m} = \sum_{s=1}^{S} \left(p_{s,1} c_{s,1} + \cdots + p_{s,M} c_{s,M} \right).$$ [1]

The markets for contingent claims open before the true state of the world is revealed. Afterwards, deliveries of the different commodities are made according to the contracts negotiated before the state is realized and then consumption occurs.

A *complete contingent claims equilibrium* (CCE) is a non-zero price function p on $\Re_+^{SM}$ and a feasible allocation $(c^1, \ldots, c^I)$ such that c^i solves

$$\max_{c^i} u_i(c^i)$$

subject to

$$p \cdot c^i \leq p \cdot \omega^i \tag{1.5}$$

for all i. The complete contingent claims equilibrium allows us to specify a competitive equilibrium under uncertainty by assuming that prices exist for consumption in each possible state of the world.

We can state the following results.

- The First Welfare Theorem: A complete contingent claims equilibrium is Pareto optimal.
- The Second Welfare Theorem: A Pareto optimal allocation can be supported as an equilibrium.

To prove the First Welfare Theorem, suppose $(c^1, c^2, \ldots, c^I, p)$ is an equilibrium which is not Pareto optimal. Then there exists an allocation

[1] Notice that $p \cdot (\alpha x + \beta y) = \alpha(p \cdot x) + \beta(p \cdot y)$ for any $\alpha, \beta \in \Re$ and $x, y \in \Re^{SM}$ so that the price function is *linear*.

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$(\hat{c}^1, \hat{c}^2, \ldots, \hat{c}^I)$ and a non-zero price vector $\hat{p}$ such that $u_i(\hat{c}^i) \geq u^i(c^i)$ for all i and $u_j(\hat{c}^j) > u^j(c^j)$ for some j. Since the utility function is strictly increasing and continuous on $\Re_+^{SM}$, then it can be easily proved that $p \cdot \hat{c}^i \geq p \cdot c^i$ for all i with strict inequality for agent j. This implies

$$p \cdot \sum_{i=1}^{I} \hat{c}^i > p \cdot \sum_{i=1}^{I} \omega^i,$$

which contradicts the feasibility of $(\hat{c}^1, \hat{c}^2, \ldots, \hat{c}^I)$.

The existence of equilibrium and the welfare theorems are discussed by Debreu [142], who provides an introduction to competitive equilibrium when the commodity space is finite-dimensional. Early proofs of the existence of a competitive equilibrium are by Arrow and Debreu [33] and McKenzie [338]. Duffie [159, 161] provides a textbook treatment.

1.1.2. *Computing the equilibrium*

What is the problem of a consumer in a contingent claims equilibrium? Let $c_s^i = (c_{s,1}^i, \ldots, c_{s,M}^i)'$. The problem in Equation (1.5) can be written as

$$\max_{\{c_s^i\}_{s=1}^S} \sum_{s=1}^{S} \pi_s^i U_i(c_s^i)$$

subject to

$$\sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m}[\omega_{s,m}^i - c_{s,m}^i] \geq 0.$$

Thus, consumer i chooses a vector of length $S \times M$ to maximize his utility subject to a budget constraint.

To analyze the consumer's problem, we make the following assumption on the utility function $U_i(c)$.

Assumption 1.1 *Let $U_i : \Re_+^{S\times M} \to \Re$ be concave, increasing, and twice continuously differentiable and*

$$\lim_{c \to 0} U_i'(c) = +\infty, \quad \lim_{c \to \infty} U_i'(c) = 0.$$

By the Kuhn–Tucker Theorem, there exists a positive Lagrange multiplier λ_i such that c^i solves the consumer's problem:

$$\max_{c^i \in \Re_+^{SM}} u_i(c^i) + \lambda_i(p \cdot \omega^i - p \cdot c^i).$$

We can write this equivalently as:

$$\max_{c^i \in \Re_+^{SM}} \sum_{s=1}^{S} \pi_s^i U_i(c_s^i) + \lambda_i \left[\sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m}\omega_{s,m}^i - p_{s,m}c_{s,m}^i \right].$$

Notice that the λ_i for $i = 1, \ldots, I$ are not state dependent. The first-order condition is

$$0 = \pi_s^i \left(\frac{\partial U_i(c^i)}{\partial c_{s,m}^i} \right) - \lambda_i p_{s,m} \quad \text{for each } s, m \text{ and } i.$$

This can be written as

$$\frac{\pi_s^i (\partial U_i(c^i)/\partial c_{s,m}^i)}{\lambda_i} = p_{s,m} \quad \text{for each } s,\ m \text{ and } i. \tag{1.6}$$

To illustrate the solution procedure, assure that the utility function U_i is separable across states s and across commodities m.

Define the functions

$$g_i(x) = (U_i')^{-1}(x).$$

These exist since marginal utility is strictly decreasing. Hence, given $\lambda_i p_{s,m}/\pi_s^i$, we can use the Implicit Function Theorem to show there is a solution

$$c_{s,m}^i = g_i(\lambda_i p_{s,m}/\pi_s^i) \tag{1.7}$$

for $s = 1, \ldots, S$, $m = 1, \ldots, M$, and $i = 1, \ldots, I$. The functions $g_i(\cdot)$ are known as the *Frisch demands*, and they express consumption allocations in terms of the product of the individual-specific Lagrange multipliers and the probability-weighted contingent claims prices.

How do we solve for the competitive equilibrium? Now go back to the initial budget constraint and substitute the solution for $c_{s,m}^i$. This yields

$$\sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m}[\omega_{s,m}^i - g_i(\lambda_i p_{s,m}/\pi_s^i)] = 0. \tag{1.8}$$

For each i, this is an equation in the unknown λ_i, given the price system. Notice that the left side is strictly increasing in λ_i. Hence, by the Implicit Function Theorem, there exists the solutions $\lambda_i^\star = h_i(p)$ for $i = 1, \ldots, I$. Given the solution for λ_i as a function of the prices p, the market-clearing conditions can be used to solve for the prices as:

$$\sum_{i=1}^{I} g_i(h_i(p)p_{s,m}/\pi_s^i) = \sum_{i=1}^{I} \omega_{s,m}^i, \quad s = 1, \ldots, S, m = 1, \ldots, M. \tag{1.9}$$

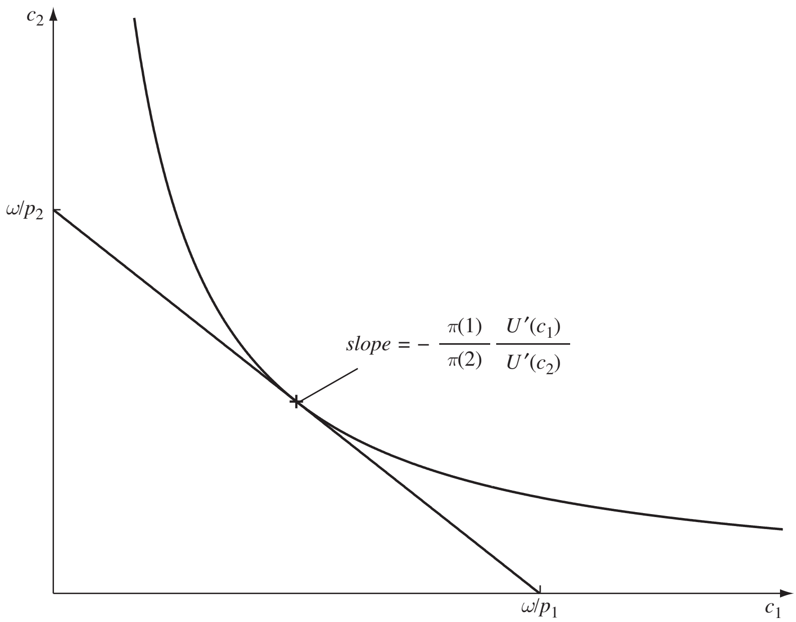

Figure 1.1. The consumer's optimum in an economy with two states

Example 1.1 Suppose that consumer i has preferences given by

$$U_i(c^i) = \frac{(c^i)^{1-\gamma} - 1}{1-\gamma} \quad \gamma \geq 0, \quad \gamma \neq 1. \tag{1.10}$$

Also specialize to the case of two states, two consumers, and one commodity per state, $S = 2$, $I = 2$ and $M = 1$, and assume that $\pi^i_s = \pi_s$.

The first-order conditions in (1.6) are now given by:

$$\pi_s U'(c^i_s) = \lambda_i p_s, \quad s = 1, 2, i = 1, 2. \tag{1.11}$$

Taking the ratios of these conditions across the two states,

$$\frac{\pi_2}{\pi_1}\frac{U'(c^i_2)}{U'(c^i_1)} = \frac{p_2}{p_1}, \quad i = 1, 2. \tag{1.12}$$

Figure 1.1 depicts the consumer's optimum.

Under the preferences given above, $U'(c) = c^{-\gamma}$ and $g_i(x) = x^{-\frac{1}{\gamma}}$. We can substitute for the utility function to evaluate the four first-order conditions in Equation (1.11) as:

$$c^i_s = (\lambda_i p_s/\pi_s)^{-\frac{1}{\gamma}}, \quad s = 1, 2, i = 1, 2. \tag{1.13}$$

Notice that there are four unknowns – $\lambda_1, \lambda_2, p_1, p_2$ – and four equations – the budget constraints for consumers $i = 1, 2$ and the market-clearing conditions for states $s = 1, 2$. Normalize the price of consumption in state 1 as $p_1 = 1$. We can substitute the solutions for c_s^i from (1.13) into the individual-specific budget constraints in (1.8) as:

$$\sum_{s=1}^{2} p_s[\omega_s^i - (\lambda_i p_s/\pi_s)^{-\frac{1}{\gamma}}] = 0, \quad i = 1, 2. \tag{1.14}$$

These equations yield the solution for λ_i as

$$\lambda_i = \left\{ \frac{\pi_1^{\frac{1}{\gamma}} + p_2^{\frac{\gamma-1}{\gamma}}(1-\pi_1)^{\frac{1}{\gamma}}}{\omega_1^i + p_2\omega_2^i} \right\}^{\gamma}, \quad i = 1, 2. \tag{1.15}$$

We can substitute these conditions into the market-clearing conditions in (1.14) to solve for the relative price of consumption in state 2, p_2.

Following this approach, the market-clearing conditions for states 1 and 2 with the solutions for c_s^i substituted in are given by

$$(\lambda_1/\pi_1)^{-\frac{1}{\gamma}} + (\lambda_2/\pi_1)^{-\frac{1}{\gamma}} = \omega_1,$$

$$(\lambda_1 p_2/(1-\pi_1))^{-\frac{1}{\gamma}} + (\lambda_2 p_2/(1-\pi_1))^{-\frac{1}{\gamma}} = \omega_2,$$

where $\omega_s = \omega_s^1 + \omega_s^2$. Now substitute for λ_1 and λ_2 using (1.15). Taking the ratio of the two market-clearing conditions yields the solution for the equilibrium price as

$$p_2 = \left(\frac{1-\pi_1}{\pi_1}\right)\left(\frac{\omega_1}{\omega_2}\right)^{\gamma}, \tag{1.16}$$

where $\omega_s = \omega_s^1 + \omega_s^2$ for $s = 1, 2$. This says that the price of consumption in state 2 relative to consumption in state 1 is a function of the ratio of the probabilities and endowments across the two states. This price also depends on consumers' willingness to substitute consumption across states described by the parameter γ. Notice that p_2 is inversely related to the probability of state 1 and the endowment in state 2. If either π_1 is high or ω_2 is large, then there will be less demand for goods delivered contingent on state 2 occurring, and p_2 will be small.

- If the aggregate endowment varies across states but the probability of state 1 equals the probability of state 2, then

$$p_2 = \left(\frac{\omega_1}{\omega_2}\right)^{\gamma}, \tag{1.17}$$

which varies monotonically with the ratio of the total endowment in each state. If $\gamma = 0$ so that consumers are indifferent between consumption in each state, then $p_2 = 1$. For values of $\gamma > 0$, p_2 decreases (increases) with γ for $\omega_1 < \omega_2$ ($\omega_1 > \omega_2$). In other words, as consumers become less willing to substitute consumption across states, the relative price of consumption across states 1 and 2 adjusts to make their demands consistent with the total endowment in each state.

- If the aggregate endowment is equal across states, $\omega_1 = \omega_2$, then

$$p_2 = \frac{1 - \pi_1}{\pi_1}. \tag{1.18}$$

Thus, the contingent claims price just equals the ratio of the probability of the two states, and it is independent of consumer preferences.[2]

We can also solve for the consumption allocations in each state by substituting for λ^i and p_2 from (1.15) and (1.16) into the Frisch demands in (1.13). However, a simpler approach is to consider the conditions in (1.12). Under the assumption for preferences, these simplify as:

$$c_1^i = \left[\frac{1 - \pi_1}{\pi_1} \frac{1}{p_2} \right]^{-\frac{1}{\gamma}} c_2^i. \tag{1.19}$$

If we substitute the expression for c_I^i obtained from equation (1.19) into the budget constraint, we obtain the solutions for c_1^i and c_2^i as

$$c_1^i = \frac{\left[(1 - \pi_1)/(\pi_1 p_2)\right]^{-\frac{1}{\gamma}} (\omega_1^i + p_2 \omega_2^i)}{\left[(1 - \pi_1)/(\pi_1 p_2)\right]^{-\frac{1}{\gamma}} + p_2}, \tag{1.20}$$

$$c_2^i = \frac{\omega_1^i + p_2 \omega_2^i}{\left[(1 - \pi_1)/(\pi_1 p_2)\right]^{-\frac{1}{\gamma}} + p_2}. \tag{1.21}$$

Notice that even if the aggregate endowment is equal across states, individual consumers' allocations will depend on the value of their endowments in states 1 and 2.

- Perfectly negatively correlated endowments across consumers.

$$\omega_1^1 = 0 \quad \omega_2^1 = 1$$
$$\omega_1^2 = 1 \quad \omega_2^2 = 0.$$

In this case, the total endowment does not vary across states so that the only uncertainty is individual-specific. Since the aggregate endowment equals one in each state, the relative price is $p_2 = (1 - \pi_1)/\pi_1$, as argued earlier. Using (1.20–1.21), we have that $c_s^1 = 1 - \pi_1$

[2] This result depends, however, on assuming that consumers have identical preferences. If consumers have different utility functions, then the equilibrium price will depend not only on the aggregate endowment in each state but also on how this endowment is split between individuals 1 and 2. In this case, p_2 will vary with γ even if the aggregate endowment is equal across states. See Exercise 2.

and $c_s^2 = \pi_1$. If the probability of state 1 is low, the individual who receives endowment in state 1 also receives low consumption in each state. Suppose $\pi_1 = 0.5$. Then the equilibrium price in each state is $p_1 = p_2 = 1$ and $c_s^i = 0.5$ for $i = 1, 2$ and $s = 1, 2$ so that consumption allocations are identical across consumers for all states.

- Positively correlated endowments across consumers.

$$\omega_1^1 = 0.5 \qquad \omega_2^1 = 1$$
$$\omega_1^2 = 1 \qquad \omega_2^2 = 2.$$

Notice that the ratio of the aggregate endowment in states 1 versus 2 equals 1/2, that is, $\omega_1/\omega_2 = 1/2$. It is straightforward to show that $c_s^2 = 2c_s^1$, that is, the second consumer always consumes twice as much as the first regardless of the probabilities of the states. Furthermore, this result does not depend on the parameter γ. The impact of changes in the endowment in this case is merely to adjust the relative price of consumption in state 2.

1.1.3. *Pareto optimal allocations*

In this section, we show the equivalence between the competitive equilibrium and Pareto optimal allocations. In later chapters, we describe how this equivalence can be exploited to characterize competitive equilibrium in a variety of settings.

Assume that U_i are strictly increasing and concave for all i. The social planner assigns weights $\eta_i \in \Re_+$ to each consumer i and chooses allocations $c^i \in \Re_+^{SM}$ for $i = 1, \ldots, I$ to maximize the weighted sum of individual utilities subject to a set of resource constraints for each state s and each commodity m:

$$\max_{c^1,\ldots,c^I} \sum_{i=1}^{I} \eta_i \sum_{s=1}^{S} \pi_s^i U_i(c_s^i) \quad \text{s. t.} \quad \sum_{i=1}^{I} c^i = \sum_{i=1}^{I} \omega^i. \tag{1.22}$$

First, notice that a feasible allocation that is Pareto optimal solves the problem in Equation (1.22) with a set of positive weights $\eta \in \Re_+^I$ and $c^1 + \cdots + c^I = \omega$. This follows as an application of the Separating Hyperplane Theorem. (See Exercise 1.) Second, notice that if $(c^1, \ldots, c^I)$ solves the problem in Equation (1.22), then it is Pareto optimal.