1 Mathematical Matching

1.1 INTRODUCTION

In 1962 a paper by David Gale and Lloyd S. Shapley[1] appeared at the RAND Corporation, whose title, "College Admissions and the Stability of Marriage," raised eyebrows. Actually, the paper dealt with a matter of some urgency.

According to Gale,[2] the paper owes its origin to an article in the *New Yorker*, dated September 10, 1960, in which the writer describes the difficulties of undergraduate admissions at Yale University. Then as now, students would apply to several universities and admissions officers had no way of telling which applicants were serious about enrolling. The students, who had every reason to manipulate, would create the impression that each university was their top choice, while the universities would enroll too many students, assuming that many of them would not attend. The whole process became a guessing game. Above all, there was a feeling that actual enrollments were far from optimal.

Having read the article, Gale and Shapley collaborated. First, they defined the concept of stable matching, and then proved that stable matching between students and universities always exists. This and further developments will be discussed in this chapter.

For simplicity, Gale and Shapley started with the unrealistic case in which there are exactly n universities and n applicants and each university has exactly one vacancy. A more realistic description of this case is a matching between men and women – hence the title of their paper.

[1] Gale, D. and Shapley, L. S. 1962. "College admissions and the stability of marriage," American Mathematical Monthly 69: 9–15.
[2] Gale, D. 2001. "The two-sided matching problem: origin, development and current issues," International Game Theory Review 3: 237–52.

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1.2 THE MATCHING PROBLEM

Consider a community of men and women where the number of men equals the number of women.

Objective: Propose a good matching system for the community.[3] To be able to propose such a system, we shall need relevant data about the community. Accordingly, we shall ask every community member to rank members of the opposite sex in accordance with his or her preferences for a marriage partner. We shall assume that no man or woman in the community is indifferent to a choice between two or more members of the opposite sex.[4] For example, if Al's list of preferences consists of Ann, Beth, Cher, and Dot, in that order, then Al ranks Ann first, Beth second, Cher third, and Dot fourth.[5] Again, we shall assume that Al is not indifferent to a choice between two or more of the four women on his list.

Example:
The men are Al, Bob, Cal, Dan.
The women are Ann, Beth, Cher, Dot.
Their list of preferences is:

Women's Preferences:

	Ann	Beth	Cher	Dot
Al	1	1	3	2
Bob	2	2	1	3
Cal	3	3	2	1
Dan	4	4	4	4

Men's Preferences:

	Ann	Beth	Cher	Dot
Al	3	4	1	2
Bob	2	3	4	1
Cal	1	2	3	4
Dan	3	4	2	1

Explanation: The numbers in the table indicate what rank a man or woman occupies in the order of preferences. For example, according to the men's ranking of the women, Al ranks Cher first, Dot second,

[3] The meaning of "good" will become clear presently.
[4] This assumption is introduced to simplify our task. In Section 1.10 we shall see how to dispense with it.
[5] If Al prefers Ann to Beth and Beth to Cher, it follows that he prefers Ann to Cher. Accordingly, we may list all his preferences in a row.

Ann third, and Beth last. And according to the women's ranking of the men, Cher ranks Bob first, Cal second, Al third, and Dan last. Thus Al ranks Cher first, while Cher ranks Al just third. If we pair them off, the match will not work out, if the first or second candidate on Cher's preference list agrees to be paired off with her.

Given everyone's preferences, can you propose a matching system for the community?

A Possible Proposal:

$$\begin{pmatrix} \text{Al} & \text{Bob} & \text{Cal} & \text{Dan} \\ | & | & | & | \\ \text{Dot} & \text{Ann} & \text{Beth} & \text{Cher} \end{pmatrix}$$
$$\quad 2 \times 2 \quad 2 \times 2 \quad 2 \times 3 \quad 2 \times 4$$

The numbers below each couple indicate what rank one member of a couple assigns to the other member. The number on the left indicates what rank the man assigns to the woman; the number on the right, what rank the woman assigns to the man. (Verify it!)

Argument for the Proposal:

(1) No members of any couple rank each other first.
(2) No members of any couple rank each other 1×2 or 2×1.
(3) The members of two couples rank each other second.
(4) Cal can be paired off with Cher or Beth, but he prefers Beth.
(5) That leaves Dan and Cher, who can be paired off.

This is indeed a possible proposal, but it is not a good one.

Cher is displeased, because she is paired off with her last choice. She can propose to Bob, but she will be turned down because she is his last choice. She will fare no better with Cal, because she is his third choice while he is paired off with his second choice. On the other hand, if Cher proposes to Al, he will be very pleased, because she is his first choice.

The proposal is rejected, because Cher and Al prefer each other to their actual mates, and one can reasonably assume that they will reject the matchmaker's proposal.

Another Possible Proposal: Let us try to pair off all the men with their first choice.

Al's first choice is Cher.
Bob's first choice is Dot.
Cal's first choice is Ann.
Dan's first choice is Dot.

We see that there is a problem: both Bob and Dan prefer Dot. We can try to pair off Dan with his second choice, Cher, but she is already paired off with Al. Will Dan's third choice work out? Dan's third choice is Ann, but she is already paired off with Cal. That leaves Dan with his last choice, Beth.

$$\begin{array}{cccc} \left(\begin{array}{cccc} \text{Al} & \text{Bob} & \text{Cal} & \text{Dan} \\ | & | & | & | \\ \text{Cher} & \text{Dot} & \text{Ann} & \text{Beth} \end{array}\right) \\ \begin{array}{cccc} 1\times 3 & 1\times 3 & 1\times 3 & 4\times 4 \end{array} \end{array}$$

Three of the four men are paired off with their first choice. Do you think this proposal will be accepted or rejected?

Still Another Possible Proposal: Now we shall try to pair off all the women with their first choice. Is it possible?

Ann's first choice is Al.
Beth's first choice is Al.
Cher's first choice is Bob.
Dot's first choice is Cal.

We see that if we pair off Ann with her first choice, Al, then Beth cannot be paired off with him too. We can pair off Beth with her second choice, Bob, but he is already paired off with Cher. And Beth's third choice, Cal, is already paired off with Dot. Beth is therefore left with her last choice, Dan.

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The new matching system is:

$$\begin{array}{c}\begin{pmatrix} \text{Ann} & \text{Beth} & \text{Cher} & \text{Dot} \\ | & | & | & | \\ \text{Al} & \text{Dan} & \text{Bob} & \text{Cal} \end{pmatrix} \\ \begin{array}{cccc} 3\times 1 & 4\times 4 & 4\times 1 & 4\times 1 \end{array}\end{array}$$

Three of the four women are paired off with their first choice. Will they accept or reject this matching system?

Beth can fight this matching. For example, she can approach Bob and suggest that they both reject this matching and form their own pair. In so doing Beth gets her second choice – better than her fourth choice – and Bob gets his third choice – better than his fourth choice. Thus, the above matching will be rejected by Beth and Bob.

Exercise: Analyze the second proposal above and see whether it can be rejected by any pair of men and women.

The first proposal was rejected, but we can turn the failed effort to our advantage. Indeed, we have learned that a matching system must satisfy the following requirement:

A matching system must be such that under it there cannot be found a man and a woman who are not paired off with each other but prefer each other to their actual mates.

Explanation: The matching system must be such that under it Ms. X cannot be paired off with Mr. x and Ms. Y cannot be paired off with Mr. y, when Ms. X prefers Mr. y to Mr. x and Mr. y prefers Ms. X to Ms. Y.

$$\begin{tikzcd} \text{Women:} & \dots & X \arrow[d, dash] \arrow[drr, <->] & \dots & Y \arrow[d, dash] & \dots \\ \text{Men:} & \dots & x & \dots & y & \dots \end{tikzcd}$$

The figure indicates the "impossible" part of the system. Specifically, the double arrow indicates that X prefers y to x and y prefers X to Y.

If couples X–x and Y–y were paired off according to the matchmaker's recommendation, then Ms. X could say to Mr. y, "You prefer me to your actual mate and I prefer you to mine. Let's leave them and pair up."

Discussion:
Will Ms. X and Mr. y pair themselves off with each other? Not necessarily! Mr. y might say, "Yes, I prefer you, X, to Y, but I prefer Z to you."
If y is lucky and Z prefers him to her actual mate, then those two can pair themselves off with each other. Otherwise, y's best choice will be X, whom he prefers to his actual mate. *Either way, the matchmaker's recommendation will not be implemented.*

Definition: A matching system is called *stable* if under it there cannot be found a man and woman who are not paired off with each other but prefer each other to their actual mates.

Example:
For simplicity, we substitute letters for names.
The men: a, b, c, d.
The women: A, B, C, D.

Preference Structure:

	A	B	C	D
a	1	2	4	2
b	2	4	2	1
c	3	1	1	3
d	4	3	3	4

	A	B	C	D
a	4	2	1	3
b	2	1	3	4
c	3	1	4	2
d	2	4	1	3

We have circled a stable matching system in the above preference structure. Later we shall learn how to find such a system.

$$\begin{array}{cccc} \begin{pmatrix} A & B & C & D \\ | & | & | & | \\ b & c & d & a \end{pmatrix} \\ 2\times 2 \quad 1\times 1 \quad 1\times 3 \quad 3\times 2 \end{array}$$

Note: The position of the circles in the two tables must be identical.

Verification: Mr. c and Mr. d are paired off with their first choice, so they need look no further. Mr. b prefers Ms. B to his actual mate, Ms. A, but Ms. B will turn him down because he is her last choice. Mr. a prefers Ms. B and Ms. C to his actual mate, D. If he proposes to B, she will turn him down because he is her second choice and she is paired off with her first choice. If he proposes to C, she too will turn him down because he is her last choice.

Remark: When no man wants to deviate from the matchmaker's recommendation, then it does not matter if a woman wants to change, because she will not find a man who will agree to cooperate with her. Thus, there is no further need to continue the verification.

1.3 EXERCISES

1. Given the following preference structure, check whether the proposed matching systems are stable. Support your answer.

Women: A, B, C. Men: a, b, c.

Women's Preferences:

	A	B	C
a	1	1	1
b	2	2	2
c	3	3	3

Men's Preferences:

	A	B	C
a	1	2	3
b	1	2	3
c	1	2	3

i. $\begin{pmatrix} A & B & C \\ | & | & | \\ a & b & c \end{pmatrix}$ ii. $\begin{pmatrix} A & B & C \\ | & | & | \\ a & c & b \end{pmatrix}$

2. Given the following preference structure, check whether the proposed matching systems are stable.

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Women: A, B, C. Men: a, b, c.

Women's Preferences:

	A	B	C
a	2	2	1
b	1	3	3
c	3	1	2

Men's Preferences:

	A	B	C
a	1	2	3
b	1	2	3
c	1	2	3

i. $\begin{pmatrix} A & B & C \\ | & | & | \\ a & b & c \end{pmatrix}$ ii. $\begin{pmatrix} A & B & C \\ | & | & | \\ c & a & b \end{pmatrix}$

iii. $\begin{pmatrix} A & B & C \\ | & | & | \\ b & a & c \end{pmatrix}$ iv. $\begin{pmatrix} A & B & C \\ | & | & | \\ b & c & a \end{pmatrix}$

3. Given the following preference structure of a community of four men and four women:

Women: A, B, C, D. Men: a, b, c, d.

Women's Preferences:

	A	B	C	D
a	1	2	4	2
b	2	4	2	1
c	4	1	1	3
d	3	3	3	4

Men's Preferences:

	A	B	C	D
a	4	2	1	3
b	2	1	3	4
c	3	1	4	2
d	2	4	1	3

(1) Is the matching system $\begin{pmatrix} A & B & C & D \\ | & | & | & | \\ b & c & d & a \end{pmatrix}$ stable?

If so, explain. If not, indicate which couple(s) will not follow the recommendation.

(2) Is the matching system $\begin{pmatrix} A & B & C & D \\ | & | & | & | \\ b & a & d & c \end{pmatrix}$ stable?

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If so, explain. If not, indicate which couple(s) will not follow the recommendation.

(3) Using the above preference structure, propose a possible matching system for this community and check its stability.

4. Given the following preference structure of a community of five women and five men:

Women: A, B, C, D, E.   Men: a, b, c, d, e.

Women's Preferences:

	A	B	C	D	E
a	5	4	3	2	1
b	1	5	4	3	2
c	2	1	5	4	3
d	3	2	1	5	4
e	4	3	2	1	5

Men's Preferences:

	A	B	C	D	E
a	1	2	3	4	5
b	5	1	2	3	4
c	4	5	1	2	3
d	3	4	5	1	2
e	2	3	4	5	1

(1) Show that the following matching systems are all stable.

i.
$$\begin{pmatrix} A & B & C & D & E \\ | & | & | & | & | \\ a & b & c & d & e \end{pmatrix}$$

ii.
$$\begin{pmatrix} A & B & C & D & E \\ | & | & | & | & | \\ e & a & b & c & d \end{pmatrix}$$

iii.
$$\begin{pmatrix} A & B & C & D & E \\ | & | & | & | & | \\ d & e & a & b & c \end{pmatrix}$$

iv.
$$\begin{pmatrix} A & B & C & D & E \\ | & | & | & | & | \\ c & d & e & a & b \end{pmatrix}$$

(2) Find another matching system with a similar structure. Is it stable?

(3) Verify that in this preference structure all preferences of the women are the reverse of the preferences of the men. For example, Ms. A is Mr. a's first preference, while Mr. a is Ms. A's last preference.

1.4 FURTHER EXAMPLES

In this section we shall present several preference structures and check whether there are any stable matching systems.

Example 1

The preference structure is:

	A	B
a	1	1
b	2	2

	A	B
a	1	2
b	1	2

There are two possible matching systems for a community of two men and two women. Let us check whether they are stable.

i.
$$\left(\begin{tikzcd} A \arrow[d, dash] & B \arrow[d, dash] \\ b & a \arrow[ul, <->] \end{tikzcd}\right)$$

ii.
$$\left(\begin{tikzcd} A \arrow[d, dash] & B \arrow[d, dash] \\ a & b \end{tikzcd}\right)$$

i. This matching system is unstable. The double arrow shows how the system can be undermined.
ii. This matching system is stable because A and a are paired off with their first choice and therefore will not deviate from the matchmaker's recommendation.

Example 2

The preference structure is:

	A	B
a	2	1
b	1	2

	A	B
a	1	2
b	1	2

The possible matching systems are:

i.
$$\left(\begin{tikzcd} A \arrow[d, dash] & B \arrow[d, dash] \\ b & a \end{tikzcd}\right)$$

ii.
$$\left(\begin{tikzcd} A \arrow[d, dash] & B \arrow[d, dash] \\ a & b \arrow[ul, <->] \end{tikzcd}\right)$$

