Quantum Groups

Algebra has moved well beyond the topics discussed in standard undergraduate texts on "modern algebra." Those books typically dealt with algebraic structures such as groups, rings and fields: still very important concepts! However, *Quantum Groups: A Path to Current Algebra* is written for the reader at ease with at least one such structure and keen to learn the latest algebraic concepts and techniques.

A key to understanding these new developments is categorical duality. A quantum group is a vector space with structure. Part of the structure is standard: a multiplication making it an "algebra." Another part is not in those standard books at all: a comultiplication, which is dual to multiplication in the precise sense of category theory, making it a "coalgebra." While coalgebras, bialgebras and Hopf algebras have been around for half a century, the term "quantum group," along with revolutionary new examples, was launched by Drinfel'd in 1986.

AUSTRALIAN MATHEMATICAL SOCIETY LECTURE SERIES

1 Introduction to Linear and Convex Programming, N. CAMERON
2 Manifolds and Mechanics, A. JONES, A. GRAY & R. HUTTON
3 Introduction to the Analysis of Metric Spaces, J. R. GILES
4 An Introduction to Mathematical Physiology and Biology, J. MAZUMDAR
5 2-Knots and their Groups, J. HILLMAN
6 The Mathematics of Projectiles in Sport, N. DE MESTRE
7 The Peterson Graph, D. A. HOLTON & J. SHEEHAN
8 Low Rank Representations and Graphs for Sporadic Groups, C. PRAEGER & L. SOICHER
9 Algebraic Groups and Lie Groups, G. LEHRER (ed.)
10 Modelling with Differential and Difference Equations, G. FULFORD, P. FORRESTER & A. JONES
11 Geometric Analysis and Lie Theory in Mathematics and Physics, A. L. CAREY & M. K. MURRAY (eds)
12 Foundations of Convex Geometry, W. A. COPPEL
13 Introduction to the Analysis of Normed Linear Spaces, J. R. GILES
14 The Integral: An Easy Approach after Kurzweil and Henstock, L. P. YEE & R. VYBORNY
15 Geometric Approaches to Differential Equations, P. J. VASSILIOU & I. G. LISLE
16 Industrial Mathematics, G. FULFORD & P. BROADBRIDGE
17 A Course in Modern Analysis and its Applications, G. L. COHEN
18 Chaos: A Mathematical Introduction, J. BANKS, V. DRAGAN & A. JONES

Quantum Groups

A Path to Current Algebra

ROSS STREET

Technical Editor:
ROSS MOORE

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/978-0-521-69524-4

First published 2007

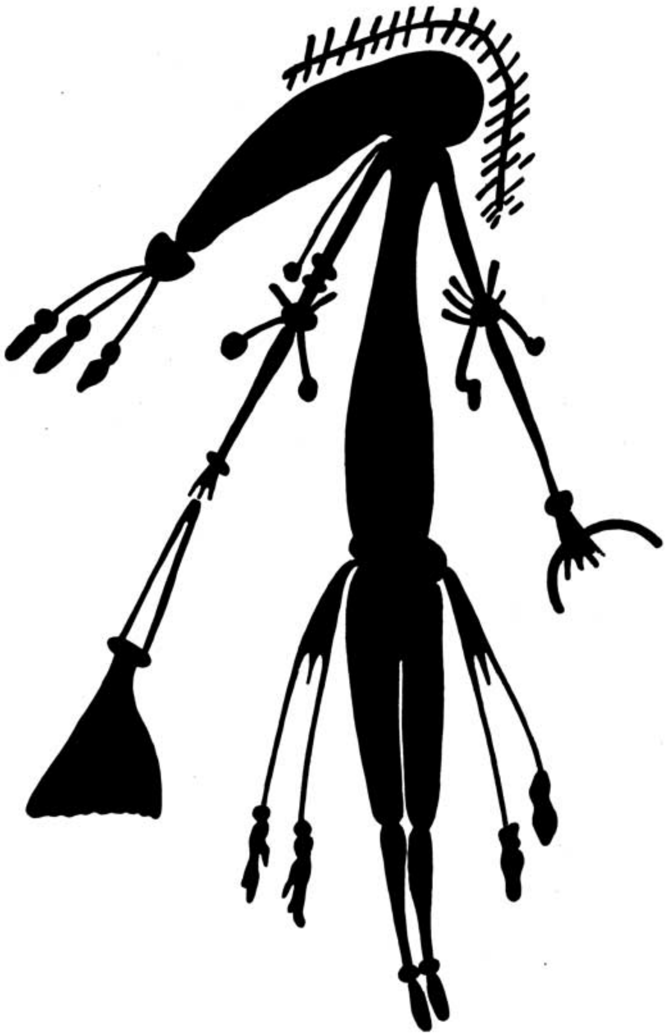
Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this publication is available from the British Library

ISBN-978-0-521-69524-4 paperback

To Oscar and Jack

Contents



Bradshaw: "Ceremonial Figure", Tassel Bradshaw Group, [Wal94, Plate 20].

Introduction

Algebra has moved well beyond the topics discussed in standard undergraduate texts on "modern algebra". Those books typically dealt with algebraic structures such as groups, rings and fields: still very important concepts! However, *Quantum Groups: A Path to Current Algebra* is written for the reader at ease with at least one such structure and keen to learn the latest algebraic concepts and techniques.

A key to understanding these new developments is categorical duality. A quantum group is a vector space with structure. Part of the structure is standard: a multiplication making it an "algebra". Another part is not in those standard books at all: a comultiplication, which is dual to multiplication in the precise sense of category theory, making it a "coalgebra". While coalgebras, bialgebras and Hopf algebras have been around for half a century, the term "quantum group", along with revolutionary new examples, was unleashed on the mathematical community by Drinfel′d [Dri87] at the International Congress in 1986. Before launching into an explanation of the duality required, I should mention here that an ordinary group gives rise to a quantum group by taking the vector space with the group as basis.

When pushed to provide formal proofs of our claims, mathematicians generally resort to set theory. We build our structures on sets and feel satisfied when we can be explicit about the elements of our constructed objects. Up to the mid twentieth century, algebraic objects were sets with selected operations which assigned elements to lists of elements. Typically, we would have binary operations which might be called addition, multiplication or Lie bracket respectively assigning a sum, product or formal commutator to each pair of elements.

In those days, the importance was recognized of dealing with the homomorphisms between algebraic structures: these were the functions which preserved the operations involved in the kind of structure at hand. The existence of a bijective homomorphism (isomorphism) between two algebraic objects meant that the two objects played the same role. So how could the literal elements be the defining ingredient? The important issue was the way the algebraic object related to others of its own kind by means of homomorphisms into it or out of it. Quite often the elements could be recaptured as homomorphisms from a particular object into the one of interest. For example, the elements of a vector space were in bijection with the linear functions from a selected one-dimensional space.

Homomorphisms into an object might therefore be called "generalized elements" of the object. However, this notion of element of the object will depend on the kind of structure we are studying since that will determine what a homomorphism is (a group homomorphism, a linear function, a ring homomorphism, or whatever).

We quite often wish to add more elements to our sets to improve the properties of the operations: as when we construct the integers from the natural numbers to obtain subtraction; or when we construct the rational numbers from the integers to obtain division; or when we construct the real polynomials from the real numbers to obtain an indeterminate. These constructions can be described explicitly as sets with operations that include the original ones. More importantly, each such construction is unique up to isomorphism with a *universal property*: each homomorphism $X \longrightarrow C$ out of the original structure X, into a set C with the extra structure, extends to a homomorphism $\hat{X} \longrightarrow C$ out of the constructed object $\hat{X}$.

In this way it was realized that knowing the homomorphisms *out of* objects determined the objects just as surely as knowing the homomorphisms *into* them did. It is natural then to call homomorphisms out of an object "generalized co-elements". Once this kind of *duality principle* is acknowledged, interesting facts appear.

Let us take a simple example purely using sets. Consider two sets X and Y. Their *cartesian product* $X \times Y$ is the set whose elements are pairs (x, y) where x lies in X and y lies in Y. We are not studying any structure on these sets except for the property of being a set. So homomorphisms in this case are merely functions. It is clear that functions $f : T \longrightarrow X \times Y$ into $X \times Y$ from a test object T are in bijection with pairs of functions (f_1, f_2) where $f_1 : T \longrightarrow X$ and $f_2 : T \longrightarrow Y$. In other words, T-elements of $X \times Y$ are in bijection with pairs consisting of a T-element of X and a T-element of Y. All that is fairly straightforward.

Now suppose that our sets X and Y have no common elements; if they are not disjoint, replace them by isomorphic sets which are. Write $X + Y$ for the union; we write $X + Y$ rather than $X \cup Y$ to emphasize that it is the disjoint union (if X and Y were finite, the number of elements of $X + Y$ would be the sum of the number of elements in X and the number in Y). A function $f : X + Y \longrightarrow T$ is determined by its restriction to X and its restriction to Y. In other words, the co-T-elements of $X + Y$ are in bijection with pairs consisting of a co-T-element of X and a co-T-element of Y.

We conclude that the constructions $X \times Y$ and $X + Y$ are duals of one another. This is not something that was stressed when we were taught the more abstract multiplication and addition of numbers in infants' school.

If we now look at vector spaces or groups X and Y, the cartesian product $X \times Y$ as sets becomes a vector space or group by means of coordinatewise operations from X and Y; again this has pairs as the generalized elements $T \longrightarrow X \times Y$. However, to obtain the dual constructions in these cases is quite different from the disjoint union of sets: in the case of vector spaces, we have that $X \times Y$ is self-dual (called direct sum and denoted by $X \oplus Y$); in the case of groups, the dual notion is rather complicated (called the free product by group theorists).

In order to formalize the way in which constructions such as these can be dual, we can use the notion of *category*. I intend to give a definition of this concept in this introduction. Before doing so, I would like to draw an analogy. It was noticed in projective plane geometry that theorems occurred in pairs: one such pair consists of Pascal's Mystic Hexagram Theorem and Brianchion's Theorem; both are about conics. Given one theorem in a pair, the other is obtained by interchanging the role of points and lines, reversing the incidence relation ("lies on" becomes "goes through"). To formally explain this duality, we abstract the notion of *projective plane*.

Here is the essence of the definition. A projective plane $\mathcal{P}$ consists of two sorts of elements: one sort called *points*, the other called *lines*. It also consists of a relation between these elements, called *incidence* (this is a rule telling when a point is incident with a line). There are some axioms which include:

1. for distinct points P and Q, there is a unique line L such that P and Q are both incident with L; and,
2. for distinct lines L and M, there is a unique point P such that P is incident with both L and M.

Any system satisfying this is a projective plane! The points do not need to look like points and the lines do not need to look like lines in any sense. Of course, we still draw pictures to help our intuition.

Now we are ready to formalize duality. Given a projective plane $\mathcal{P}$, we obtain a projective plane $\mathcal{P}^{\mathrm{rev}}$ whose points are the lines of $\mathcal{P}$, whose lines are the points of $\mathcal{P}$, and whose incidence relation is the reverse of that of $\mathcal{P}$. Notice that axioms (1) and (2) for $\mathcal{P}^{\mathrm{rev}}$ are respectively axioms (2) and (1) for $\mathcal{P}$. This means that, if we prove a theorem about all projective planes, then the dual theorem is automatically true by applying the original theorem to $\mathcal{P}^{\mathrm{rev}}$.

It turns out that there are not too many interesting theorems assuming only axioms (1) and (2). A further axiom based on a theorem of Pappus can be added and the system remains self-dual. In fact, conics can be defined using an idea of Steiner, and Pascal's Theorem can be proved. Let us now discontinue discussion of this analogy and return to the formalization of the duality at hand.

A *category* $\mathcal{A}$ consists of two sorts of elements: one sort called *objects*, the other called *morphisms* (or *arrows*). It also consists of three functions. The first function assigns to each morphism f a pair (A, B) of objects in which case A is called the *domain* (or *source*) of f while B is called the *codomain* (or *target*) of f; the notations $f : A \longrightarrow B$ and $A \xrightarrow{f} B$ are used. The second function assigns to each object A a morphism $1_A : A \longrightarrow A$ called the *identity morphism* of A. A pair (f, g) of morphisms is called *composable* when the codomain of f is equal to the domain of g. The third function assigns to each composable pair (f, g) of morphisms, a morphism $g \circ f$, called the *composite* of f and g, whose domain is that of f and whose codomain is that of g. There are two axioms:

1. if (f, g) and (g, h) are composable pairs of morphisms then $(h \circ g) \circ f = h \circ (g \circ f)$; and,
2. if $f : A \longrightarrow B$ is a morphism then $f \circ 1_A = f = 1_B \circ f$.

The standard argument shows that identity morphisms are unique. The notation $\mathcal{A}(A, B)$ (or $\mathrm{Hom}_{\mathcal{A}}(A, B)$) is used for the set of all morphisms in $\mathcal{A}$ from A to B.

There is a category **Set** whose objects are sets, morphisms are functions, and composition is the usual composition of functions. There is a category $\mathbf{Vect}_{\mathbf{k}}$ whose objects are vector spaces over a fixed field $\mathbf{k}$ and morphisms are linear functions; composition is as usual. Similarly we have a category whose objects are groups and a category whose objects are rings.

However, there are categories whose objects do not look like sets and whose morphisms do not look like functions. For example, there is a category whose objects are integers, whose morphisms are pairs (m, n) of integers such that the domain of (m, n) is m and the codomain is the product mn; a pair $((m, n), (r, s))$ of morphisms is composable when $mn = r$ and the pair's composite is (m, ns).

Now to duality. Given a category $\mathcal{A}$, there is a category $\mathcal{A}^{\mathrm{op}}$ whose objects are the objects of $\mathcal{A}$, and morphisms are the morphisms of $\mathcal{A}$; however, the domain of a morphism is its codomain in $\mathcal{A}$ while its codomain in $\mathcal{A}^{\mathrm{op}}$ is its domain in $\mathcal{A}$. A pair (g, f) of morphisms is composable in $\mathcal{A}^{\mathrm{op}}$ if and only if (f, g) is composable in $\mathcal{A}$; its composite $f \circ g$ in $\mathcal{A}^{\mathrm{op}}$ is the composite $g \circ f$ in $\mathcal{A}$. We call $\mathcal{A}^{\mathrm{op}}$ the *dual* or *opposite* of the category $\mathcal{A}$.

Perhaps it helps to say that $\mathcal{A}^{\mathrm{op}}$ is the category obtained from $\mathcal{A}$ by reversing arrows: a morphism $f : A \longrightarrow B$ in $\mathcal{A}$ is precisely a morphism $f : B \longrightarrow A$ in $\mathcal{A}^{\mathrm{op}}$. Admittedly, if the objects of $\mathcal{A}$ look like sets (that is, are sets with some structure), the same is true of $\mathcal{A}^{\mathrm{op}}$; but the same cannot be said for morphisms that are functions, since formally reversed functions can scarcely be thought of as functions.

The duality between cartesian product and disjoint union can now be made precise. In a category $\mathcal{A}$, a *product* for two objects A and B consists of an object $A \times B$ and two morphisms $p_1 : A \times B \longrightarrow A$, $p_2 : A \times B \longrightarrow B$ (called *projections*) with the following "universal" property: for all objects T and morphisms $a : T \longrightarrow A$, $b : T \longrightarrow B$, there exists a unique morphism $T \longrightarrow A \times B$, denoted by (a, b), such that $p_1 \circ (a, b) = a$ and $p_2 \circ (a, b) = b$. This means that T-elements of $A \times B$ are in bijection with pairs consisting of a T-element of A and a T-element of B.

A morphism $h : C \longrightarrow D$ in a category $\mathcal{A}$ is called a *right inverse* for a morphism $k : D \longrightarrow C$ when $k \circ h = 1_C$; we also say that k is a *left inverse* for h. A morphism h is *invertible* (or an *isomorphism*) when it has both a left and right inverse; in this case, a familiar argument shows that the left and right inverse agree and this common morphism is unique, being called the *inverse* of h and denoted by h^{-1}. If there exists an invertible morphism $C \longrightarrow D$ then we say C and D are *isomorphic* and write $C \cong D$. In a category, we think of isomorphic objects as being essentially the same. Any two products of two objects A and B can be proved, by an easy argument, to be isomorphic.

Now we have our duality between cartesian product and disjoint union of sets: cartesian product is the product in the category **Set** while disjoint union is the product in the category $\mathbf{Set}^{\mathrm{op}}$.

We can give an even simpler example. An object K of a category $\mathcal{A}$ is called *terminal* when, for all objects A of $\mathcal{A}$, there is precisely one morphism $A \longrightarrow K$. The singleton set 1 is terminal in the category **Set** while the empty set $\varnothing$ is terminal in $\mathbf{Set}^{\mathrm{op}}$.

Any concept defined for all categories $\mathcal{A}$ has a dual concept which is the same concept translated to $\mathcal{A}^{\mathrm{op}}$: the prefix "co-" is used. So a product in $\mathcal{A}^{\mathrm{op}}$ is called a *coproduct* in $\mathcal{A}$. A terminal object in $\mathcal{A}^{\mathrm{op}}$, under this system, would be called a *coterminal object* in $\mathcal{A}$; but it is also called an *initial object* of $\mathcal{A}$.

In the spirit of category theory itself, we should consider appropriate morphisms of categories. These are called *functors*. A functor $F : \mathcal{A} \longrightarrow \mathcal{X}$ between categories $\mathcal{A}$ and $\mathcal{X}$ consists of two functions. The first assigns to each object A of $\mathcal{A}$ an object FA of $\mathcal{X}$. The second function assigns to each morphism $f : A \longrightarrow B$ of $\mathcal{A}$ a morphism $Ff : FA \longrightarrow FB$ of $\mathcal{X}$. There are two axioms:

1. $F1_A = 1_{FA}$ for all objects A of $\mathcal{A}$; and,
2. $F(g \circ f) = Fg \circ Ff$ for all composable pairs (f, g) in $\mathcal{A}$.

It is easy to see that functors preserve invertibility of morphisms: in fact they take inverses to inverses.

Let us look at a couple of examples of functors.

- Each object T of a category $\mathcal{A}$ determines a functor $R_T = \mathcal{A}(T, \_) : \mathcal{A} \longrightarrow \mathbf{Set}$ called the functor *represented by* T; the elements of $R_T A = \mathcal{A}(T, A)$ are the morphisms $a : T \longrightarrow A$ in $\mathcal{A}$ (that is, the T-elements of A), while the function $R_T f : R_T A \longrightarrow R_T B$ takes such an a to $f \circ a$.

- Suppose K is an object of $\mathcal{A}$ for which a product $K \times A$ exists (and is chosen) for all objects A. There is a functor $F = K \times \_ : \mathcal{A} \longrightarrow \mathcal{A}$ defined on objects by $FA = K \times A$ and on morphisms by $Ff = K \times f$ where $K \times f = (p_1, f \circ p_2) : K \times A \longrightarrow K \times B$.

Categories were invented not only to formalize duality but to formalize the concept of "naturality" in mathematics. The idea was that a natural transformation should be one that involves no ad hoc choices. For example, if V is a vector space and V^* is the vector space of linear functions from V into the base field $\mathbf{k}$, there is a natural linear function $V \longrightarrow V^{**}$ which takes $v \in V$ to the linear function $e_v : V^* \longrightarrow \mathbf{k}$ defined by evaluation at v. However, any linear function $V^{**} \longrightarrow V$ that depends on a choice of basis for V should not be natural.

Suppose $F : \mathcal{A} \longrightarrow \mathcal{X}$ and $G : \mathcal{A} \longrightarrow \mathcal{X}$ are functors between the same categories. A *natural transformation* $\theta : F \longrightarrow G$ is a function. The function assigns to each object A of $\mathcal{A}$ a morphism $\theta_A : FA \longrightarrow GA$ of $\mathcal{X}$. There is a single axiom: for each morphism $f : A \longrightarrow B$,

$$Gf \circ \theta_A = \theta_B \circ Ff .$$

There is an obvious componentwise composition of natural transformations. This defines a category $[\mathcal{A}, \mathcal{X}]$, called a *functor category*, where the objects are functors $F : \mathcal{A} \longrightarrow \mathcal{X}$ and the morphisms are natural transformations. A *natural isomorphism* is an invertible morphism in the functor category. A functor $F : \mathcal{A} \longrightarrow \mathbf{Set}$ is called *representable* when it is isomorphic to R_T for some object T; such a T is called a *representing object* for F. For example, the functor $U : \mathbf{Vect}_{\mathbf{k}} \longrightarrow \mathbf{Set}$, which takes each vector space to its underlying set and each linear function to that function, is representable: we have $U \cong R_{\mathbf{k}}$ since the linear functions from the field $\mathbf{k}$ to a vector space V are in natural bijection with elements of V. Many constructions in mathematics are designed to provide representing objects for interesting functors.

Let us look at a couple of examples of natural transformations:

- Suppose $F : \mathcal{A} \longrightarrow \mathbf{Set}$ is a functor and T is an object of $\mathcal{A}$. Each element x of FT determines a natural transformation $\hat{x} : R_T \longrightarrow F$ defined by $\hat{x}_A(a) = (Fa)(x)$. The *Yoneda Lemma* states that this defines a bijection $FT \cong [\mathcal{A}, \mathbf{Set}](R_T, F)$. The inverse bijection is even easier: it takes the natural transformation $\theta : R_T \longrightarrow F$ to the element $\theta_T(1_T)$ of FT.

- Suppose $h : K \longrightarrow L$ is a morphism of a category $\mathcal{A}$ in which products of pairs of objects exist. Then we obtain a natural transformation $f \times \_ : K \times \_ \longrightarrow L \times \_$ whose value at the object A is the morphism $f \times A = (f \circ p_1, p_2) : K \times A \longrightarrow L \times A$.

Modern algebra in the sense of the first half of the twentieth century dealt with sets equipped with operations. Soon after, the idea of co-operations crept into mathematics. The notion of coalgebra is dual to algebra. This is the main concept in this book.

Now I turn to the book's contents. Chapter 1 gives precise definitions of monoids and groups; the axioms are expressed in terms of diagrams ready to be imported to a general category. This importation is carried out in Chapter 2 where we provide the important example of 2×2 matrices in readiness for the quantum version. A duality between geometry and algebra is explained. In Chapter 3, we describe the quantum general linear group of 2×2 matrices as a coalgebra. This comes from lectures by Manin in Montréal. Chapter 4 is about modules over rings; we find it natural to take a 2-sided point of view so that our basic module M has a left action by a ring R and a right action by a ring S which compatibly interact. Chapter 5 concerns finitely generated, projective modules under the mysterious name of "Cauchy modules". It turns out that F. W. Lawvere noticed a concept in enriched category theory which has Cauchy sequences as an example; when interpreted for additive categories, it leads to modules that are finitely generated and projective. Chapter 6 discusses algebras, Lie algebras and the Poincaré–Birkhoff–Witt Theorem. Chapter 7 is about coalgebras and bialgebras. A coalgebra is a vector space with a comultiplication. A bialgebra is an algebra which is also a coalgebra subject to a compatibility condition. The dual vector space of a coalgebra is an algebra, however, the usual dual of an algebra need not naturally be a coalgebra. In Chapter 8 we describe Sweedler's modification (see [Swe69] and [Abe80]) of the dual of an algebra which is a coalgebra.

In Chapter 9, we look at Hopf algebras. These should be thought of as generalized groups. An important part of group theory is the theory of their representations: these are modules over the group ring. In Chapter 10 we look at modules over bialgebras. Then, in Chapter 11, we move to use categories more seriously. We discuss categories equipped with an abstract tensor product: monoidal or tensor categories. We discuss examples involving braids. A deep example, not treated here, is the subject of the paper [JS95]. An important property of the tensor product $U \otimes V$ of vector spaces is that it represents the vector space $[V, W]$ of linear functions from V to W:

$$\mathbf{Vect}_k(U \otimes V, W) \cong \mathbf{Vect}_k(U, [V, W]) .$$

In Chapter 12, this idea is lifted to arbitrary tensor categories. Examples from knot theory are provided.

The Yang–Baxter equation, from the branch of physics called statistical mechanics, had a major influence on the new examples of Hopf algebras called quantum groups. In Chapter 13, an algebraic concept of Yang–Baxter operator, which makes sense in any tensor category, is explained. A family of examples from linear algebra is provided in Chapter 14.

In Chapter 15, the notion of monoid is lifted to the level of generality at which "algebras" and "coalgebras" become precise categorical duals. For the first time, I believe, in a text at this level, emphasis is placed on "2-cells" between monoid morphisms, providing the student with a gentle introduction to higher-dimensional category theory.

Each bialgebra has a tensor category of representations. This correspondence is a modern formulation of Tannaka–Krein duality. The treatment of this topic in Chapter 16 makes use of the 2-dimensional structure of monoids from Chapter 15. There is by now a vast literature on Tannaka duality. We satisfy ourselves with a sketch in Chapter 17 of an application to construct universally a Hopf algebra from a bialgebra.

Finally, in Chapter 18, the example of Chapter 3 is revisited in the light of what has been learned. There are exercises at the end of several chapters. Solutions to most of these are provided in Chapter 19.

Acknowledgements ...

Many ideas presented here are my version of joint research with André Joyal. I consider myself very fortunate to have experienced such exciting collaboration.

I would like to thank the students and staff who attended the original course in the first half of 1990 at Macquarie University. I am very grateful to Paddy McCrudden (as a Vacation Scholar under my supervision in January–February 1995) for his careful reading of my handwritten notes, and particularly for his systematic checking of the exercises.

It is a pleasure to acknowledge the support of grants from the Australian Research Council during the preparation of this work.

... the typing process

Many of the chapters were carefully typed by Elaine Vaughan. Technology moved ahead incredibly from that word-processor available in our Department in 1990. Typesetting with TEX and LATEX was begun by Ross Moore and continued by several post-doc and graduate students; namely Sam Williams, Ross Talent (now deceased) and Simon Byrne. Each added further exercises and solutions, as these were encountered in lecture courses.

Most important in this was the use of XY-pic[1] to produce the commutative diagrams that appear throughout this book, and which play such an integral part in the visualization and understanding of much of the material. None of these diagrams has been imported as an image built using other software. All, including the braids, tangles, and 2-cell diagrams, are

[1]Originally written by Kristoffer Rose; extended and enhanced by Rose and Moore for mathematical applications and higher quality output. The XY-pic package and documentation is now included with all TEX and LATEX distributions.

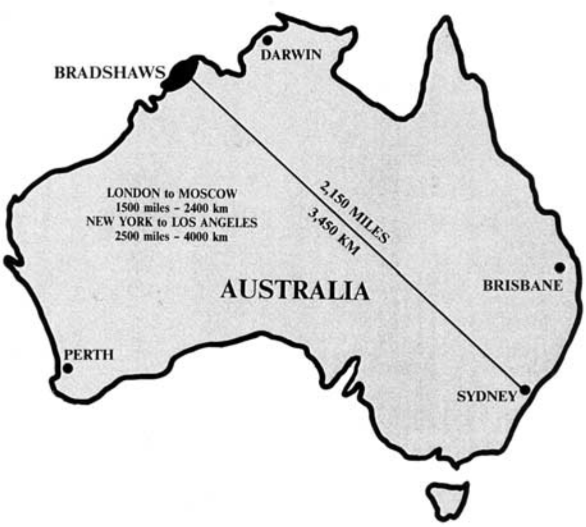
specified within the LaTeX source using the XY-pic package syntax. Indeed the syntax and coding to handle curves and 2-cells was written in 1993–94 by Ross Moore, specifically for use with this book. Since then the XY-pic package has become a useful tool for presenting diagrammatic material in Category Theory and other branches of mathematics, computer science and linguistics.

As an application of Ross Moore's work with the LaTeX2HTML translation software, an earlier version of the present manuscript was made available via the "world-wide web", now known as the internet. In that form it was used as a source of lecture notes for courses at Macquarie.

A great deal of credit is also due to Simon Byrne (as a Vacation Scholar in January–February 2005) for finishing off the typing of exercises and for assembling the manuscript into a form ready to submit as a proposal for the Australian Mathematical Society Lecture Series. With this go-ahead, the final version of the manuscript, complete with up-to-date Bibliography, Index, front-matter and filler images was prepared by Ross Moore, who is acknowledged here as being the Technical Editor for this monograph.

… the illustrations

The illustrations appearing at the end of some chapters are reproduced from Grahame Walsh, *Bradshaws* [Wal94]. I am very grateful to the Bradshaw Foundation and Edition Limitée for consenting to their inclusion. The original coloured rock paintings, which the silhouettes trace, are the work of Australian people living as much as 50 millennia before our time.



These paintings have been mentioned already in the mathematico-scientific literature, in connection with knots and braids; *viz.*

> How old are knots? It has been suggested that the Stone Age should be called the Age of String. The extraordinary tasselled figures photographed and described by G. L. Walsh in *Bradshaws: Ancient Rock Paintings of Western Australia* (Edition Limitée, 1994) have been suggested to be 50,000 years old. Knots have been intimately linked with the development of humans, through weapons, fishing, hunting, clothing, housing, boating and a myriad of other ways.
>
> The metaphor of knots is found throughout literature, and knots and interlacing are featured in many forms of art.

Ronald Brown, review of: "The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots" by Colin C. Adams (W. H. Freeman 1994), appeared in *Nature*, Vol. 371 (13 October 1994).

Suggested Further Reading

[JS91b] André Joyal and Ross Street. An introduction to Tannaka duality and quantum groups. In *Category Theory (Como, 1990)*, volume 1488 of *Lecture Notes in Mathematics*, pages 413–492. Springer, Berlin, 1991. **MR1173027**

[Kas95] Christian Kassel. *Quantum Groups*, volume 155 of *Graduate Texts in Mathematics*. Springer, New York, 1995. **MR1321145**

[Maj95] Shahn Majid. *Foundations of Quantum Group Theory*. Cambridge University Press, Cambridge, 1995, (paperback, 2000). **MR1381692**

[SS93] Steven Shnider and Shlomo Sternberg. *Quantum Groups: From CoAlgebas to Drinfel'd Algebras*. Graduate Texts in Mathematical Physics, II. International Press, Cambridge, MA, 1993. **MR1287162**

[Yet01] David N. Yetter. *Functorial Knot Theory*, volume 26 of *Series on Knots and Everything*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001. **MR1834675**