GROUPS AND SEMIGROUPS: CONNECTIONS AND CONTRASTS

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1   Introduction

Group theory and semigroup theory have developed in somewhat different directions in the past several decades. While Cayley's theorem enables us to view groups as groups of permutations of some set, the analogous result in semigroup theory represents semigroups as semigroups of functions from a set to itself. Of course both group theory and semigroup theory have developed significantly beyond these early viewpoints, and both subjects are by now integrally woven into the fabric of modern mathematics, with connections and applications across a broad spectrum of areas.

Nevertheless, the early viewpoints of groups as groups of permutations, and semigroups as semigroups of functions, do permeate the modern literature: for example, when groups act on a set or a space, they act by permutations (or isometries, or automorphisms, etc.), whereas semigroup actions are by functions (or endomorphisms, or partial isometries, etc.). Finite dimensional linear representations of groups are representations by invertible matrices, while finite dimensional linear representations of semigroups are representations by arbitrary (not necessarily invertible) matrices. The basic structure theories for groups and semigroups are quite different — one uses the ideal structure of a semigroup to give information about the semigroup for example — and the study of homomorphisms between semigroups is complicated by the fact that a congruence on a semigroup is not in general determined by one congruence class, as is the case for groups.

Thus it is not surprising that the two subjects have developed in somewhat different directions. However, there are several areas of modern semigroup theory that are closely connected to group theory, sometimes in rather surprising ways. For example, central problems in finite semigroup theory (which is closely connected to automata theory and formal language theory) turn out to be equivalent or at least very closely related to problems about profinite groups. Linear algebraic monoids have a rich structure that is closely related to the subgroup structure of the group of units, and this has interesting connections with the well developed theory of (von Neumann) regular semigroups. The theory of inverse semigroups (i.e., semigroups of partial one-one functions) is closely tied to aspects of geometric and combinatorial group theory.

In the present paper, I will discuss some of these connections between group theory and semigroup theory, and I will also discuss some rather surprising contrasts between the theories. While I will briefly mention some aspects of finite semigroup theory, regular semigroup theory, and the theory of linear algebraic monoids, I

will focus primarily on the theory of inverse semigroups and its connections with geometric group theory.

For most of what I will discuss, there is no loss of generality in assuming that the semigroups under consideration have an identity — one can always just adjoin an identity to a semigroup if necessary — so most semigroups under consideration will be monoids, and on occasions the group of units (i.e., the group of invertible elements of the semigroup) will be of considerable interest.

2   Submonoids of Groups

It is perhaps the case that group theorists encounter semigroups (or monoids) most naturally as submonoids of groups. For example, if P is a submonoid of a group G such that $P \cap P^{-1} = \{1\}$, then the relation $\leq_P$ on G defined by $g \leq_P h$ iff $g^{-1}h \in P$ is a left invariant partial order on G. This relation is also right invariant iff $g^{-1}Pg \subseteq P$ for all $g \in G$, and it is a total order iff $P \cup P^{-1} = G$. Note that $g \in P$ iff $1 \leq_P g$. Every left invariant partial order on G arises this way. One says that (G, P) is a partially ordered group with positive cone P. One may note that the partial order has the property that for all $g \in G$ there exists some $p \in P$ such that $g \leq_P p$ iff $G = PP^{-1}$, i.e., iff G is the group of (right) quotients of P. The study of ordered groups is well over a hundred years old, and I will not attempt to survey this theory here.

The question of embeddability of a semigroup (monoid) in a group is a classical question that has received a lot of attention in the literature. Clearly a semigroup must be cancellative if it is embeddable in a group. It is easy to see that commutative cancellative semigroups embed in abelian groups, in fact such a semigroup embeds in its group of quotients in much the same way as an integral domain embeds in a field. For non-commutative semigroups, the situation is far more complicated. One useful condition in addition to cancellativity that guarantees embeddability of a semigroup S in a group is the Ore condition. A semigroup S satisfies the *Ore condition* if any two principal right ideals intersect, i.e., $sS \cap tS \neq \emptyset$ for all $s, t \in S$. (In the language of many subsequent authors in the group theory literature, s and t have at least one *common multiple* for each $s, t \in S$: in the language of classical semigroup theory, one says that S is *left reversible*). The following well known result was essentially proved by Ore in 1931: a detailed proof may be found in Volume 1, Chapter 1 of the book by Clifford and Preston [30], which is a standard reference for basic classical results and notation in semigroup theory. There is an obvious dual result involving right reversible semigroups and groups of left quotients.

Theorem 1 *A cancellative semigroup satisfying the Ore condition can be embedded in a group. In fact a cancellative semigroup P can be embedded in a group $G = PP^{-1}$ of (right) quotients of P if and only if P satisfies the Ore condition.*

As far as I am aware, the first example of a cancellative semigroup that is not embeddable in a group was provided by Mal'cev in 1937 [85]. Necessary and sufficient conditions for the embeddability of a semigroup in a group were provided by

Mal'cev in 1939 [86]. Mal'cev's conditions are countably infinite in number and no finite subset of them will suffice to ensure embeddability of a semigroup in a group. A similar set of conditions, with a somewhat more geometric interpretation, was provided by Lambek in 1951. Chapter 10 of Volume 2 of Clifford and Preston [30] provides an account of the work of Mal'cev and Lambek and a description of the relationship between the two sets of conditions.

The question of when a monoid with presentation $P = Mon\langle X : u_i = v_i\rangle$ embeds in a group has been studied by many authors, and has received attention in the contemporary literature in group theory. Clearly such a monoid embeds in a group if and only if it embeds in the group with presentation $G = Gp\langle X : u_i = v_i\rangle$. Here the u_i, v_i are *positive* words, i.e., $u_i, v_i \in X^*$, where X^* denotes the free monoid on X. We allow for the possibility that some of the words u_i or v_i may be empty, (i.e., the identity of X^*). Also, we use the notation $Mon\langle X : u_i = v_i\rangle$ for the monoid presented by the set X of generators and relations of the form $u_i = v_i$ to distinguish it from the group $Gp\langle X : u_i = v_i\rangle$ or the semigroup $Sgp\langle X : u_i = v_i\rangle$ with the same set of generators and relations. From an algorithmic point of view, the embeddability question is undecidable, as are many such questions about semigroup presentations or group presentations, since the property of being embeddable in a group is a Markov property (see Markov's paper [94]).

It is perhaps worth observing that being embeddable in a group is equivalent to *being* a group for special presentations where all defining relations are of the form $u_i = 1$. Recall that the *group of units* of a monoid P is the set

$$U(P) = \{a \in P : ab = ba = 1 \text{ for some } b \in P\}.$$

Proposition 1 *Let P be a monoid with presentation of the form $P = Mon\langle X : u_i = 1, \ i = 1, \ldots, n\rangle$, where each letter of X is involved in at least one of the relators u_i. Then P is embeddable in a group if and only if it is a group.*

Proof Suppose that P is embeddable in a group G, and consider a relation $u_i = 1$ in the set of defining relations of P. If $u_i = x_1x_2\ldots x_n$ with each $x_j \in X$, then clearly x_1 is the inverse of $x_2\ldots x_n$ in G, so x_1 is in the group of units of P and $x_2\ldots x_nx_1 = 1$ in P also. It follows that x_2 is in the group of units of P, and similarly each x_j must be in the group of units of P. Since this holds for each relator u_i, and since each letter in X is involved in some such relator, every letter of X (i.e., every generator of P) must lie in the group of units of P, so P is a group. □

Remark We remark at this point that the word problem for one-relator monoids with a presentation of the form $M = Mon\langle X : u = 1\rangle$ was solved by Adian [2]. However the word problem for semigroups with one defining relation of the form $S = Sgp\langle X : u = v\rangle$ where both u and v are non-empty words in X^* remains open, as far as I am aware. There has been considerable work done on the one-relator semigroup problem in general (see for example, the papers by Adian and Oganessian [3], Guba [54], Lallement [75], Watier [144], and Zhang [148]). Later in

this paper, I will indicate how this problem is related to the membership problem
for certain submonoids of one-relator groups.

Despite the difficulties in deciding embeddability of a semigroup in a group in
general, there are many significant results in the literature that show that monoids
(semigroups) with particular presentations may be embedded in the corresponding
groups. Perhaps the first such general result along these lines was obtained by
Adian [1].

Let P be a semigroup with presentation $P = Sgp\langle X : u_i = v_i, \ i = 1, \ldots, n \rangle$,
where u_i, v_i are strictly positive (i.e., non-empty) words. The *left graph* for this
presentation is the graph with set X of vertices and with an edge from x to y if
there is a defining relation of the form $u_i = v_i$ where x is the first letter of u_i and y
is the first letter of v_i. The *right graph* is defined dually. The semigroup P is called
an *Adian semigroup* and the corresponding group $G = Gp\langle X : u_i = v_i \rangle$ is called an
Adian group if both the left graph and the right graph are cycle-free (i.e., if both
graphs are forests). Of course a presentation is regarded as cycle-free if it contains
no defining relations.

Theorem 2 (Adian [1]) *Any Adian semigroup embeds in the corresponding Adian
group.*

Remmers [118] gave a geometric proof of this using semigroup diagrams, and
Stallings [132] gave another proof using a graph theoretic lemma. Sarkisian [124]
apparently gave a proof of the decidability of the membership problem for an Adian
semigroup P in the corresponding Adian group G, and used this to solve the word
problem for Adian groups: unfortunately there appears to be a gap in the proof
in [124]. Adian's results have been extended in different directions in the work
of several authors (see, for example, the papers by Kashintsev [70], Guba [53],
Krstic [74], and Kilgour [72], where various small cancellation conditions are used
to study embeddability of semigroups in groups).

We remark that in general an Adian group G is not the group of quotients of the
corresponding Adian semigroup P. For example, if we consider the presentation
$P = Sgp\langle a, b : ab = b^2a \rangle$, then P is an Adian semigroup whose associated Adian
group is the Baumslag–Solitar group $G = BS(1, 2)$. Not all elements of G belong
to PP^{-1}, for example $a^{-1}ba \notin PP^{-1}$. However, every element of G can be written
as a product of two elements of PP^{-1} — see Stallings [133] for a discussion of this
example. Stallings shows that if P is an Adian semigroup, then PP^{-1} is a quasi-
pregroup for the corresponding Adian group G (that is, if $q_1, q_2, \ldots, q_n \in PP^{-1}$,
$n > 1$ and $q_i q_{i+1} \notin PP^{-1}$ for all i, then $q_1 q_2 \ldots q_n \neq 1$ in G).

As a second large class of important examples of semigroups that are embeddable
in groups, we turn to a brief discussion of braid groups and Artin groups. The braid
monoid on $n + 1$ strings is the monoid P_n with presentation

$$Mon\langle x_1, x_2, \ldots, x_n : x_i x_j = x_j x_i \text{ if } |i - j| > 1,$$
$$x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ if } i = 1, \ldots, n - 1 \rangle.$$

The corresponding group with the same presentation as a group, is the braid group B_n on $n+1$ strands. Braid groups have been the object of intensive study in the literature (see for example the influential book of Birman [21], and many subsequent papers dealing with braids and their connection to other areas of mathematics). Braid monoids play a prominent role in the theory of braid groups. Garside [46] showed that such monoids satisfy the Ore condition, in fact the principal right ideals form a lattice: for each $a, b \in B_n$, there exists $c \in B_n$ such that $aB_n \cap bB_n = cB_n$, and also B_n is cancellative. Thus, by Ore's theorem we have:

Theorem 3 (Garside) *For each n, the braid monoid P_n embeds in the braid group B_n, and B_n is the group of quotients of P_n. Furthermore, the principal right ideals of P_n form a semilattice (in fact a lattice) under intersection.*

This result was simultaneously generalized by Brieskorn and Saito [23] and by Deligne [41] to Artin groups and monoids of finite type. Recall that a group G is called an *Artin group* and the corresponding monoid is called an *Artin monoid* if it is presented by a set X subject to relations of the form $prod(x, y; m_{x,y}) = prod(y, x; m_{x,y})$ if $m_{x,y} < \infty$. (Here $m_{x,x} = 1$ and $m_{x,y} = m_{y,x} \in \{2, 3, \ldots, \infty\}$ for $x, y \in X$, and $prod(x, y; m_{x,y})$ stands for the alternating word $xyxy\ldots$ of length $m_{x,y}$). An Artin group (monoid) is said to be *of finite type* if the corresponding Coxeter group is finite. These results were further generalized by Dehornoy and Paris [37] to a class of groups known as *Garside groups*, and were generalized further by Dehornoy [36] to a class of groups that admit a *thin* group of fractions, and to a group that arises in the study of left self distributivity and its connection to mathematical logic (see the book by Dehornoy [35] for full details about this). Many properties of braid groups, Artin groups of finite type, Garside groups and the more general groups considered by Dehornoy are proved by a deep study of the associated monoid of positive elements. We refer to the papers of Dehornoy cited above for further references and details. These groups admit a presentation where every relation is of the form $xu = yv$ for $x \neq y \in X$ and admit one such relation for each pair $x \neq y \in X$, so their left graphs are in fact cliques. Thus this class of groups is very different from the class of Adian groups. We also refer to the recent papers by Paris [109] and Godelle and Paris [50] where the authors solve Birman's conjecture [22] for braid groups and right angled Artin groups by studying the emebedding of singular braid monoids (Artin monoids) in the corresponding groups.

Several authors have studied the question of embeddability of general Artin monoids in Artin groups: for example special cases of this question have been considered by Charney [25] and Cho and Pride [28]. Much additional information about embeddability of semigroups in groups may be found in the paper by Cho and Pride. Paris [108] has established the following deep general result about Artin groups and Artin monoids.

Theorem 4 (Paris) *Every Artin monoid embeds in the corresponding Artin group.*

It is worth remarking that while Artin groups of finite type and the more general groups considered by Dehornoy et al. are groups of fractions of their corresponding

monoids of positive elements, this is not the case for Artin groups in general. The fact that braid groups, Artin groups of finite type, thin groups of fractions, etc., are all groups of fractions of their positive monoids leads to a fast algorithm for solving the word problem for such groups — they have quadratic isoperimetric inequality and admit an automatic structure. However, the word problem for Artin groups in general remains open, as far as I am aware.

I will close this section with brief mention of another prominent example of a monoid that embeds in its group of fractions. Recall that the Thompson group F can be defined by the presentation $F = Gp\langle x_0, x_1, \ldots : x_n x_k = x_k x_{n+1} \text{ for } k < n\rangle$.

This group has appeared in numerous settings, having been originally introduced by R. Thompson (see [101]) as a group that acts naturally on bracketed expressions by moving the brackets, i.e., by applying the associative law. We refer the reader to the monograph by Cannon, Floyd and Parry [24] for an introduction to the Thompson group F and some of its many connections with other areas of mathematics. The Thompson monoid is the monoid defined by the same relations as those that define F as a group. The following result appears to be well known.

Theorem 5 *The Thompson monoid embeds in the Thompson group F. Furthermore, F is the group of fractions of the Thompson monoid.*

A closely related group is the group G_{LD} introduced by Dehornoy [34] to describe the geometry of the left self-distributive law $x(yz) = (xy)(xz)$. See Dehornoy's book [35] for further information and deep connections with mathematical logic. It is known that the group G_{LD} is the group of quotients of an appropriate submonoid of this group, but a presentation for that submonoid seems to be unknown.

There are several ways to associate an **inverse monoid** with the situation when P is a monoid that embeds in a group G with the same presentation. We recall first some basic definitions and facts about regular and inverse monoids, and some of the rather extensive structure theory for such monoids.

3   Regular and Inverse Monoids

A monoid M is called a *(von Neumann) regular* monoid, if for each $a \in M$ there exists some element $b \in M$ such that $a = aba$ and $b = bab$. Such an element b is called *an inverse* of a (it is not necessarily unique). Note that regular monoids have in general lots of *idempotents*: if b is an inverse of a in M then ab and ba are both idempotents of M, i.e., $(ab)^2 = ab$ and $(ba)^2 = ba$ (and in general $ab \neq ba$). A monoid M is called an *inverse monoid* if for each $a \in M$ there exists a unique inverse (denoted by a^{-1}) in M such that

$$a = aa^{-1}a \quad \text{and} \quad a^{-1} = a^{-1}aa^{-1}.$$

Equivalently, M is inverse iff it is regular and the idempotents of M commute. Thus if M is an *inverse* monoid then the idempotents of M form a commutative idempotent semigroup with respect to the product in M. Since a commutative idempotent semigroup may be viewed as a lower semilattice (with meet operation equal to the product), we normally refer to such semigroups as *semilattices*. We

will consistently denote the set of idempotents of a monoid M by $E(M)$. Thus if M is an inverse monoid, then $E(M)$ is a submonoid of M that is a semilattice, referred to as the *semilattice of idempotents* of M. Every inverse monoid M comes equipped with a *natural partial order* defined by

$$a \leq b \text{ if and only if } a = eb \text{ for some idempotent } e \in M.$$

If $e = e^2$ is an idempotent of a monoid M, then the set

$$H_e = \{a \in M : ae = ea = e \text{ and } \exists b \in M \text{ such that } ab = ba = e\}$$

is a *subgroup* of M with identity e (i.e., it forms a group with identity e relative to the multiplication in M). Clearly H_e is the largest subgroup of M with identity e, and $H_e \cap H_f = \emptyset$ if $e \neq f$. It is also clear that $H_1 = U(M)$, the group of units of the monoid M. The subgroups H_e, $e \in E(M)$, are referred to as the *maximal subgroups* of M. The semilattice of idempotents and the maximal subgroups of an inverse monoid M give us a good deal of information about M, but do not by any means determine the structure of M: in general, not all elements of an inverse monoid need belong to subgroups of the monoid.

A standard example of a regular monoid is the *full transformation monoid* on a set X, which consists of all functions from X to itself with respect to composition of functions. The group of units of this monoid is of course the symmetric group on X. Idempotents in this monoid consist of functions that are identity maps on their ranges, and the maximal subgroup corresponding to such an idempotent is isomorphic to the symmetric group on the range of the map. Every semigroup can be embedded in an appropriate full transformation monoid (see Clifford and Preston [30], Volume 1).

Another standard example of a regular semigroup is the *full linear monoid* $M_n(k)$ of $n \times n$ matrices with entries in a field k, with respect to matrix multiplication. The group of units of $M_n(k)$ is the general linear group $GL_n(k)$. From elementary linear algebra we know that an idempotent matrix of rank r is similar to the diagonal matrix with block diagonal identity matrix I_r in the top left hand corner and zeroes elsewhere. The group $GL_n(k)$ acts by conjugation on the set of idempotent matrices, and the orbits of this action consist of idempotent matrices of a fixed rank. The idempotent matrices in $M_n(k)$ may be identified with pairs of opposite parabolic subgroups of $GL_n(k)$. The maximal subgroup corresponding to an idempotent matrix of rank r is isomorphic to the general linear group $GL_r(k)$. Of course the idempotents of $M_n(k)$ do not form a subsemigroup if $n > 1$. We refer to Okninski's book [106] for a detailed description of this monoid, and to Putcha's book [114] for an introduction to the elegant theory of linear algebraic monoids. A linear algebraic monoid is regular if and only if its group of units is a reductive group: the subgroup structure of the group of units of a linear algebraic monoid provides very detailed information about the structure of the monoid (see [114]).

Clearly every group is an inverse monoid (in fact groups are just regular monoids with precisely one idempotent), and every semilattice E is an inverse monoid with $e^{-1} = e$ and with $H_e = \{e\}$, for all $e \in E$. A more enlightening example of an inverse monoid is the *symmetric inverse monoid* on a set X, denoted by $SIM(X)$.

The monoid $SIM(X)$ is the monoid of all partial one-one maps (i.e., one-one maps from subsets of X to subsets of X) with respect to multiplication of partial maps: if α and β are partial one-one maps, then $\alpha\beta(x) = \alpha(\beta(x))$ whenever this makes sense, i.e., if $x \in dom(\beta)$ and $\beta(x) \in dom(\alpha)$. The group of units of $SIM(X)$ is obviously the symmetric group (the group of permutations) on X, and the idempotents of $SIM(X)$ are the identity maps on subsets of X, so the semilattice of idempotents of $SIM(X)$ is isomorphic to the lattice of subsets of X. The empty subset corresponds to the zero of $SIM(X)$: a product $\alpha\beta$ of two partial one-one maps on X is zero (the empty map) if $range(\beta) \cap dom(\alpha) = \emptyset$. The maximal subgroup corresponding to the identity map on the subset Y of X is the symmetric group on Y. The natural partial order on $SIM(X)$ is defined by domain restriction of a partial one-one map, i.e., $\alpha \le \beta$ iff $dom(\alpha) \subseteq dom(\beta)$ and $\alpha = \beta|_{dom(\alpha)}$. (I note that the definition of $SIM(X)$ given here is the dual of the usual definition found in many books on semigroup theory, where functions are traditionally written on the right rather than the left.)

Symmetric inverse monoids are in a sense generic inverse monoids.

Theorem 6 (Vagner–Preston) *Every inverse monoid embeds in a suitable symmetric inverse monoid.*

Thus inverse monoids may be viewed as monoids of partial one-one maps, in much the same way as groups may be viewed as groups of permutations. Inverse monoids arise naturally whenever one encounters partial one-one maps throughout mathematics. For example, the Vagner–Preston theorem has been extended by Barnes [10] and Duncan and Paterson [44] to show that every inverse monoid embeds as a monoid of partial isometries of some Hilbert space, and from this point of view, inverse monoids play an increasingly important role in the theory of operator algebras (see the book by Paterson [110] for an introduction to the role of inverse monoids in this theory). The book by Petrich [111] or the more recent book by Lawson [76] provide an account of the general theory of inverse monoids and some of their connections with other areas of mathematics.

Another natural class of examples of inverse monoids arises in connection with submonoids of groups. Note that any submonoid P of a group must be a left and right cancellative monoid. Let P be any left cancellative monoid. The left regular representation $a \to \lambda_a$, where $\lambda_a : x \to ax$ for all $a, x \in P$, defines an embedding of P into the symmetric inverse monoid $SIM(P)$, since each map λ_a is clearly a partial one-one map on P with domain P and range aP. The submonoid of $SIM(P)$ generated by the image of P in this embedding into $SIM(P)$ is an inverse monoid, referred to as the *(left) inverse hull* $I_l(P)$ of P. Of course there is a dual inverse monoid $I_r(P)$ that arises from the right regular representation of a right cancellative monoid P. This is the most obvious way in which inverse monoids arise in connection with submonoids of groups. I will discuss some other ways of associating inverse monoids with submonoids of groups later in this paper.

The ideal structure of a monoid provides a basic tool for beginning to study the structure of the monoid. It will be convenient to introduce some standard terminology along these lines. There are five equivalence relations, known as the

Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ that play a prominent role in the theory. For a monoid M we define

$$\mathcal{R} = \{(a,b) \in M \times M : aM = bM\},$$
$$\mathcal{L} = \{(a,b) \in M \times M : Ma = Mb\},$$
$$\mathcal{J} = \{(a,b) \in M \times M : MaM = MbM\},$$
$$\mathcal{H} = \mathcal{R} \cap \mathcal{L},$$

and $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$ (the join of $\mathcal{R}$ and $\mathcal{L}$ in the lattice of equivalence relations on M).

The corresponding equivalence classes containing $a \in M$ are denoted by R_a, L_a, J_a, H_a and D_a respectively. Clearly $\mathcal{H} \subseteq \mathcal{R}$, $\mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$. It is a fortunate fact in semigroup theory that the equivalence relations $\mathcal{R}$ and $\mathcal{L}$ commute, i.e., $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$, and it follows that $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. Thus $a\mathcal{D}b$ in M iff $\exists c \in M$ such that $a\,\mathcal{R}\,c\,\mathcal{L}\,b$ iff $\exists d \in M$ such that $a\,\mathcal{L}\,d\,\mathcal{R}\,b$. For an inverse monoid M it is easy to see that $a\,\mathcal{R}\,b$ iff $aa^{-1} = bb^{-1}$ and $a\,\mathcal{L}\,b$ iff $a^{-1}a = b^{-1}b$. It is a well-known fact that if P is a cancellative monoid, then P embeds as the $\mathcal{R}$-class R_1 of 1 in its right inverse hull $I_r(P)$ and as the $\mathcal{L}$-class L_1 of 1 in its left inverse hull $I_l(P)$.

It is informative to provide an explicit description of the Green's relations in the full linear monoid $M_n(k)$. If A and B are two matrices in $M_n(k)$, then

$A\,\mathcal{R}\,B$ iff $A\,GL_n(k) = B\,GL_n(k)$ iff $Col(A) = Col(B)$,

$A\,\mathcal{L}\,B$ iff $GL_n(k)\,A = GL_n(k)\,B$ iff $Nul(A) = Nul(B)$,

$A\,\mathcal{J}\,B$ iff $GL_n(k)\,A\,GL_n(k) = GL_n(k)\,B\,GL_n(k)$ iff $rank(A) = rank(B)$, and $\mathcal{J} = \mathcal{D}$.

Furthermore, for each fixed $r \leq n$, the group $GL_n(k)$ acts transitively by left multiplication [resp., right multiplication] on the set of $\mathcal{R}$-classes of $M_n(k)$ [resp., $\mathcal{L}$-classes of $M_n(k)$] within the $\mathcal{J}$-class J_r consisting of the matrices of rank r. In addition, if Y_r denotes the set of all matrices of rank r that are in reduced row echelon form and if X_r is the set of transposes of elements of Y_r, then the $\mathcal{R}$-classes of $M_n(k)$ in the $\mathcal{J}$-class J_r are in one-one correspondence with the matrices in X_r, and the $\mathcal{L}$-classes of $M_n(k)$ in the $\mathcal{J}$-class J_r are in one-one correspondence with the matrices in Y_r. Every matrix in J_r has a unique decomposition of the form XGY with $X \in X_r$, $Y \in Y_r$ and $G \in GL_n(k)$. Proofs of all of these facts and much additional interesting information about full linear monoids may be found in [106].

It is a well known fact in semigroup theory that if a $\mathcal{D}$-class contains a regular element, then every element of that $\mathcal{D}$-class is regular. The structure of regular $\mathcal{D}$-classes is very nice: for example, all $\mathcal{H}$-classes within the $\mathcal{D}$-class are of the same cardinality, an $\mathcal{H}$-class is a maximal subgroup iff it contains an idempotent, and two maximal subgroups contained in the same $\mathcal{D}$-class are isomorphic. A $\mathcal{D}$-class is regular iff it contains an idempotent, and if a is a regular element of M, then every inverse of a lies in the $\mathcal{D}$-class D_a. A product ab lies in the $\mathcal{H}$-class $R_a \cap L_b$ iff $L_a \cap R_b$ contains an idempotent. Proofs of these facts may be found in any standard book on semigroup theory, for example [30].

A semigroup S is called *simple* [resp., *bisimple*] if it contains just one $\mathcal{J}$-class [resp., $\mathcal{D}$-class]. A semigroup with just one $\mathcal{H}$-class is a group. A semigroup S with

a zero element 0 is called 0-*simple* if $S^2 \neq 0$ and S has only two $\mathcal{J}$-classes ($\{0\}$ and $S - \{0\}$). Such a semigroup is called 0-*bisimple* if it has just two $\mathcal{D}$-classes ($\{0\}$ and $S - \{0\}$). The structure of finite simple and 0-simple semigroups was determined by Suschkewitsch in 1928 [142]. This was extended by Rees [117] in 1940 to a class of simple [resp., 0-simple] semigroups known as *completely simple* [resp., *completely 0-simple*] semigroups. We refer to [30] for an account of this important work.

Bisimple inverse monoids may be constructed from right cancellative monoids whose principal left ideals form a semilattice. The following theorem was proved by Clifford [29] in 1953.

Theorem 7 (Clifford, 1953) *Let M be a bisimple inverse monoid with identity 1 and let $R = R_1$, the $\mathcal{R}$-class of 1. Then R is a right cancellative monoid and the principal left ideals of R form a semilattice under intersection, i.e., for each $a, b \in R$, there exists $c \in R$ such that $Ra \cap Rb = Rc$. Conversely, let R be a right cancellative monoid in which the intersection of any two principal left ideals is a principal left ideal. Then the (right) inverse hull of R is a bisimple inverse monoid and the $\mathcal{R}$-class of 1 in this monoid is a submonoid that is isomorphic to R.*

Again, there is an obvious dual construction of bisimple inverse monoids from left cancellative monoids whose principal right ideals form a semilattice. We thus have the following corollary of Garside's theorem (Theorem 3) and Clifford's theorem (Theorem 7): the result extends to Artin groups of finite type, Garside groups, thin groups of quotients, etc.

Corollary 1 *The inverse hull of the braid monoid B_n is a bisimple inverse monoid.*

If S is an inverse semigroup, then the natural partial order induces a homomorphism from S onto its maximal group homomorphic image. For S an inverse semigroup and $a, b \in S$ we define an equivalence relation σ on S by $a \, \sigma \, b$ iff $\exists c \in S$ such that $c \leq a$ and $c \leq b$. It is easy to see that σ is a *congruence* on S (i.e., it is compatible with respect to multiplication on both sides), so the set of σ-classes of S forms a semigroup S/σ with respect to the obvious multiplication, and there is a natural map (which we denote again by σ) from S onto S/σ. It is straightforward to see that S/σ is a group, the maximal group homomorphic image of S.

The inverse semigroup S is called *E-unitary* if the inverse image under σ of the identity of the group S/σ consists just of the semilattice $E(S)$ of idempotents of S. Equivalently, S is E-unitary if $a \geq e$ and $e \in E(S)$ implies $a \in E(S)$. E-unitary inverse semigroups play an essential role in the theory of inverse semigroups. Their structure has been determined by McAlister [96] by means of a group acting by order automorphisms on a partially ordered set with an embedded semilattice. Furthermore, McAlister proved [97] that if S is any inverse semigroup, then there is some E-unitary inverse semigroup T and an *idempotent-separating* homomorphism f from T onto S (a homomorphism $f : T \rightarrow S$ is called "idempotent-separating" if distinct idempotents of T are mapped to distinct idempotents of S). In this situation, we refer to T as an *E-unitary cover* of S over the group G, where G is the maximal group homomorphic image of T. We refer to Lawson [76] for