CAMBRIDGE

Cambridge University Press
978-0-521-69469-8 - Groups St Andrews 2005, Volume 1
Edited by C. M. Campbell, M. R. Quick, E. F. Robertson and G. C. Smith
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ASPECTS OF INFINITE PERMUTATION GROUPS

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Abstract

Until 1980, there was no such subject as 'infinite permutation groups', according to the Mathematics Subject Classification: permutation groups were assumed to be finite. There were a few papers, for example [10, 62], and a set of lecture notes by Wielandt [72], from the 1950s.

Now, however, there are far more papers on the topic than can possibly be summarised in an article like this one.

I shall concentrate on a few topics, following the pattern of my conference lectures: the random graph (a case study); homogeneous relational structures (a powerful construction technique for interesting permutation groups); oligomorphic permutation groups (where the relations with other areas such as logic and combinatorics are clearest, and where a number of interesting enumerative questions arise); and the Urysohn space (another case study). I have preceded this with a short section introducing the language of permutation group theory, and I conclude with briefer accounts of a couple of topics that didn't make the cut for the lectures (maximal subgroups of the symmetric group, and Jordan groups).

I have highlighted a few specific open problems in the text. It will be clear that there are many wide areas needing investigation! I have also included some additional references not referred to in the text.

1 Notation and terminology

This section contains a few standard definitions concerning permutation groups. I write permutations on the right: that is, if g is a permutation of a set Ω, then the image of α under g is written αg.

The *symmetric group* $\mathrm{Sym}(\Omega)$ on a set Ω is the group consisting of all permutations of Ω. If Ω is infinite and c is an infinite cardinal number not exceeding Ω, the *bounded symmetric group* $\mathrm{BSym}_c(\Omega)$ consists of all permutations moving fewer than c points; if $c = \aleph_0$, this is the *finitary symmetric group* $\mathrm{FSym}(\Omega)$ consisting of all *finitary* permutations (moving only finitely many points). The *alternating group* $\mathrm{Alt}(\Omega)$ is the group of all even permutations, where a permutation is even if it moves only finitely many points and acts as an even permutation on its support.

Assuming the Axiom of Choice, the only non-trivial normal subgroups of $\mathrm{Sym}(\Omega)$ for an infinite set Ω are the bounded symmetric groups and the alternating group.

A *permutation group* on a set Ω is a subgroup of the symmetric group on Ω. As noted above, we denote the image of α under the permutation g by αg. For the most part, I will be concerned with the case where Ω is countably infinite.

The permutation group G on Ω is said to be *transitive* if for any $\alpha, \beta \in \Omega$, there exists $g \in G$ with $\alpha g = \beta$. For $n \le |\Omega|$, we say that G is *n-transitive* if, in its induced action on the set of all n-tuples of distinct elements of Ω, it is transitive: that is, given two n-tuples $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ of distinct elements, there exists $g \in G$ with $\alpha_i g = \beta_i$ for $i = 1, \ldots, n$. In the case when Ω is infinite, we say that G is *highly transitive* if it is n-transitive for all positive integers n. If a permutation group is not highly transitive, the maximum n for which it is n-transitive is its *degree of transitivity*. (Of course, the condition of n-transitivity becomes stronger as n increases.)

The bounded symmetric groups and the alternating group are all highly transitive. We will see that there are many other highly transitive groups!

The subgroup $G_\alpha = \{g \in G : \alpha g = \alpha\}$ of G is the *stabiliser* of α. Any transitive action of a group G is isomorphic to the action on the set of right cosets of a point stabiliser, acting by right multiplication.

The permutation group G on Ω is called *semiregular* (or *free*) if the stabiliser of any point of Ω is the trivial subgroup; and G is *regular* if it is semiregular and transitive. Thus, a regular action of G is isomorphic to the action on itself by right multiplication.

A transitive permutation group G on Ω is *imprimitive* if there is a G-invariant equivalence relation on Ω which is not trivial (that is, not the relation of equality, and not the relation with a single equivalence class Ω). If no such relation exists, then G is *primitive*. A G-invariant equivalence relation is called a *congruence*; its equivalence classes are *blocks of imprimitivity*, and the set of blocks is a *system of imprimitivity*. A block or system of imprimitivity is non-trivial if the corresponding equivalence relation is. A non-empty subset B of Ω is a block if and only if $B \cap B^g = B$ or $\emptyset$ for all $g \in G$.

A couple of simple results about primitivity:

Proposition 1.1 (a) *The transitive group G on Ω is primitive if and only if G_α is a maximal proper subgroup of G for some (or every) $\alpha \in \Omega$.*

(b) *A 2-transitive group is primitive.*

(c) *The orbits of a normal subgroup of a transitive group G form a system of imprimitivity. Hence, a non-trivial normal subgroup of a primitive group is transitive.*

In connection with the last part of this result, we say that a transitive permutation group is *quasiprimitive* if every non-trivial normal subgroup is transitive.

Let G be a group, and S a subset of G. The *Cayley graph* $\mathrm{Cay}(G, S)$ is the (directed) graph with vertex set G, having directed edges (g, sg) for all $g \in G$ and $s \in S$. If $1 \notin S$, then this graph has no loops; if $s \in S \Rightarrow s^{-1} \in S$, then it is an undirected graph (that is, whenever (g, h) is an edge, then also (h, g) is an edge, so we can regard edges as unordered pairs). It is easy to see that $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Most importantly, G, acting on itself by right multiplication, is a group of automorphisms of $\mathrm{Cay}(G, S)$. In this situation,

G acts regularly on the vertex set of $\mathrm{Cay}(G, S)$. Conversely, if a graph Γ admits a group G as a group of automorphisms acting regularly on the vertices, then Γ is isomorphic to a Cayley graph for G. (Choose a point $\alpha \in \Omega$, and take S to be the set of elements s for which $(\alpha, \alpha s)$ is an edge.)

2 The random graph

2.1 B-groups

According to Wielandt [71], a group A is a *B-group* if any primitive permutation group G which contains the group A acting regularly is doubly transitive; that is, if any overgroup of A which kills all the A-invariant equivalence relations necessarily kills all the non-trivial A-invariant binary relations. The letter B stands for Burnside, who showed that a cyclic group of prime-power but not prime order is a B-group. The proof contained a gap which was subsequently fixed by Schur, who invented and developed Schur rings for this purpose. The theory of Schur rings (or S-rings) is connected with many topics in representation theory, quasigroups, association schemes, and other areas of mathematics; historically, it was an important source of ideas in these subjects. The theory of S-rings and its connection with representation theory is described in Wielandt's book.

Following the classification of finite simple groups, much more is known about B-groups, since indeed it is known that primitive groups are comparatively rare. For example, the set of numbers n for which there exists a primitive group of degree n other than S_n and A_n has density zero [21], and hence the set of orders of non-B-groups has density zero. (However, there are non-B-groups of every prime power order, and a complete description is not known.)

One could then ask:

> *Are there any infinite B-groups?*

Remarkably, no example of an infinite B-group is known. One of the most powerful nonexistence theorems is the following result. A *square-root set* in a group X is a set of the form

$$\sqrt{a} = \{x \in X : x^2 = a\};$$

it is *non-principal* if $a \neq 1$. A slightly weaker form of this theorem (using a stronger form of the condition, and concluding only that A is not a B-group) was proved by Graham Higman; the form given here is due to Ken Johnson and me [20].

Theorem 2.1 *Let A be a countable group with the following property:*

> *A cannot be written as the union of finitely many translates of non-principal square-root sets together with a finite set.*

Then A is not a B-group. More precisely, there exists a primitive but not 2-transitive group G which contains a regular subgroup isomorphic to each countable group satisfying this condition.

Note that the condition of Theorem 2.1 is not very restrictive: any countable abelian group of infinite exponent satisfies the condition; and for any finite or countable group A, the direct product of A with an infinite cyclic group satisfies it, so the group G of the theorem embeds every countable group as a semiregular subgroup.

2.2 The Erdős–Rényi Theorem

The group G of the last subsection is the automorphism group of the remarkable *random graph* R (sometimes known as the *Rado graph*). In the rest of this section we consider this graph and some of its properties.

The reason for the name is the following theorem of Erdős and Rényi [28]:

Theorem 2.2 *There exists a countable (undirected simple) graph R with the property that, if a graph X on a fixed countable vertex set is chosen by selecting edges independently at random with probability $1/2$ from the unordered pairs of vertices, then* $\mathrm{Prob}(X \cong R) = 1$.

Proof The proof depends on the following graph property denoted by $(*)$:

> Given two finite disjoint sets U, V of vertices, there exists a vertex z joined to every vertex in U and to no vertex in V.

Now the theorem is immediate from the following facts:

1. $\mathrm{Prob}(X \text{ satisfies } (*)) = 1$;
2. Any two countable graphs satisfying $(*)$ are isomorphic.

To prove (1), we have to show that the event that $(*)$ fails is null. This event is the union of countably many events, one for each choice of the pair (U, V) of sets; so it is enough to show that the probability that no z exists for given U, V is zero. But the probability that n given vertices $z_1, \ldots, z_n$ fail to satisfy $(*)$, where $|U \cup V| = k$, is $(1 - 1/2^k)^n$, which tends to 0 as $n \to \infty$.

Claim (2) is proved by a simple back-and-forth argument. Any partial isomorphism between two countable graphs satisfying $(*)$ can be extended so that its domain or range contains one additional point. Proceeding back and forth for countably many steps, starting with the empty partial isomorphism, we obtain the desired isomorphism. □

This is a fine example of a non-constructive existence proof: almost all countable graphs have property $(*)$, but no example of such a graph is exhibited! There are many constructions: here are a few. In each case, to show that the graph is isomorphic to R, we have to verify that $(*)$ holds.

1. Let M be a countable model of the Zermelo–Fraenkel axioms of set theory. (The existence of such a model is *Skolem's paradox*.) Thus M consists of a countable set carrying a binary relation $\in$. Form a graph by symmetrising this relation: that is, $\{x, y\}$ is an edge if $x \in y$ or $y \in x$. This graph is isomorphic to R. [Indeed, not all the ZF axioms are required here; it is enough to have the empty set, pair set, union, and foundation axioms.]

2. As a specialisation of the above, we have Rado's model of *hereditarily finite set theory* (satisfying all the ZF axioms except the axiom of infinity): the set of elements is $\mathbb{N}$, and $x \in y$ if and only if the xth digit in the base 2 expansion of y is equal to 1. This is the construction of R in [54].
3. Let $\mathbb{P}_1$ be the set of primes congruent to 1 mod 4; for $p, q \in \mathbb{P}_1$, join p to q if and only if p is a quadratic residue mod q. (By quadratic reciprocity, this relation is symmetric.) Verification of $(*)$ uses Dirichlet's Theorem on primes in arithmetic progressions.

2.3 Properties of R

A graph Γ is said to be *homogeneous* if every isomorphism between finite induced subgraphs extends to an automorphism of Γ.

Theorem 2.3 *R is homogeneous.*

Proof Given an isomorphism between finite induced subgraphs, the back-and-forth method of the preceding proof extends it to an automorphism. □

We now consider various properties of the group $G = \mathrm{Aut}(R)$. By homogeneity, G acts transitively on the vertices, (oriented) edges, and (oriented) non-edges of R; so, acting on the vertex set, it is a transitive group with (permutation) rank 3 (that is, three orbits on ordered pairs of vertices). Moreover, G is primitive on the vertices. For suppose that $\equiv$ is a congruence on the vertex set which is not the relation of equality, so that there are distinct vertices v, v' with $v \equiv v'$. Suppose that $\{v, v'\}$ is an edge (the argument in the other case is similar). Since G is transitive on edges, it follows that for every edge $\{w, w'\}$, we have $w \equiv w'$. Now let u, u' be non-adjacent vertices. Choosing $U = \{u, u'\}$ and $V = \emptyset$ in property $(*)$, we find a vertex z joined to u and u'. Thus $u \equiv z \equiv u'$, so $u \equiv u'$. Thus $\equiv$ is the universal congruence.

Thus, a group which can be embedded as a regular subgroup of G (that is, a group A for which some Cayley graph is isomorphic to R) is not a B-group. Now Theorem 2.1 is proved by showing that, if A is a group satisfying the hypotheses of the theorem, then a random Cayley graph for A is isomorphic to R with probability 1, and hence that almost all Cayley graphs for A are isomorphic to R.

As an example of the proof technique, I show:

Proposition 2.4 *R has $2^{\aleph_0}$ non-conjugate cyclic automorphisms.*

Proof If a graph Γ has a cyclic automorphism σ, then we can index the vertices by integers so that σ acts as the cyclic shift $x \mapsto x+1$. Then the graph Γ is determined by the set $S = \{x \in \mathbb{Z} : x > 0, x \sim 0\}$, where $\sim$ is the adjacency relation in the graph Γ. Indeed, $\Gamma = \mathrm{Cay}(\mathbb{Z}, S \cup (-S))$. Furthermore, it is an easy calculation to show that, if the same graph Γ has two cyclic automorphisms σ_1 and σ_2, giving rise to sets S_1 and S_2, then $S_1 = S_2$ if and only if σ_1 and σ_2 are conjugate in $\mathrm{Aut}(\Gamma)$. So the theorem will be proved if we can show that there are $2^{\aleph_0}$ different sets S for which the resulting graph Γ is isomorphic to R.

We do this by choosing the elements of S independently at random from the positive integers, and showing that the probability that $(*)$ fails is zero. Suppose that we are testing, for a given pair (U, V), whether there is a vertex z "correctly joined". Of course, we must exclude the elements of $U \cup V$ from consideration. We must discard all z for which we have already decided about the membership of any element of the form $|z - u|$ or $|z - v|$ in S, for $u \in U$ and $v \in V$: there are finitely many such. We also discard any element z for which an equation of the form $z - w_1 = w_2 - z$ holds, for $w_1, w_2 \in U \cup V$; again there are only finitely many such z. So for all but finitely many z, the events $z \sim w$ for $w \in U \cup V$ are independent, and the probability that such a z is correctly joined is non-zero.

Now the argument proceeds as in the proof of the Erdős–Rényi theorem. Since the event $\Gamma \cong R$ has probability 1, it certainly has cardinality $2^{\aleph_0}$. □

An example of a countable group for which no Cayley graph is isomorphic to R is

$$A = \langle a, b : b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

Every element of the form $a^i b^j$ in this group with j odd is a square root of b^2, and the remaining elements form a translate of this square root set. Hence, in a Cayley graph for A, we cannot find a point joined to 1 and b but not to b^2 or b^3.

Problem 2.5 Is the above group A a B-group?

2.4 Properties of Aut(R)

A number of properties of the group $G = \mathrm{Aut}(R)$ are known:

(a) It has cardinality $2^{\aleph_0}$. (This follows from Proposition 2.4, or from (c) below.)

(b) It is simple [64].

(c) It has the strong small index property (see below) [34, 18].

Truss proved that G is simple by showing that, given any two non-identity elements $g, h \in G$, it is possible to write h as the product of five conjugates of $g^{\pm 1}$. Subsequently [66] he improved this: only three conjugates of g are required. This is best possible.

A permutation group on a countable set X is said to have the *small index property* if every subgroup of index less than $2^{\aleph_0}$ contains the pointwise stabiliser of a finite set; it has the *strong small index property* if every such subgroup lies between the pointwise and setwise stabilisers of a finite set. Hodges *et al.* [34] showed that G has the small index property, and Cameron [18] improved this to the strong small index property.

Truss [65] found all the cycle structures of automorphisms of R. In particular, R has cyclic automorphisms. (This is the assertion that the infinite cyclic group is a regular subgroup of G.)

A number of subgroups of G with remarkable properties have been shown to exist. Here are two examples, due to Bhattacharjee and Macpherson [6, 7]. The first settles a question of Peter Neumann.

Theorem 2.6 *There exist automorphisms f, g of R such that*

(a) *f has a single cycle on R, which is infinite,*

(b) *g fixes a vertex v and has two cycles on the remaining vertices (namely, the neighbours and non-neighbours of v),*

(c) *the group $\langle f, g\rangle$ is free and is transitive on vertices, edges, and non-edges of R, and each of its non-identity elements has only finitely many cycles on R.*

This theorem is proved by building the permutations f and g as limits of partial maps constructed in stages. Of course the existence of automorphisms satisfying (a) and (b) follows from Truss' classification of cycle types, but much more work is required to achieve (c).

Theorem 2.7 *There is a locally finite group G of automorphisms of R which acts homogeneously (that is, any isomorphism between finite subgraphs can be extended to an element of G).*

This theorem uses a result of Hrushovski [35] on extending partial automorphisms of graphs.

Various other subgroups acting homogeneously on R can be constructed. For example, using either of the explicit descriptions (numbers 2 and 3) of R given in Section 2.2, the group of automorphisms given by recursive (or primitive recursive) functions (on $\mathbb{N}$ or $\mathbb{P}_1$ respectively) acts homogeneously.

We noted that $\mathrm{Aut}(R)$ embeds all finite or countable groups. On a similar note, Bonato *et al.* [8] showed that the endomorphism monoid of R embeds all finite or countable monoids.

2.5 Topology

I now turn to properties of G as a topological group. There is a natural topology on the symmetric group of countable degree, the topology of pointwise convergence: a basis of neighbourhoods of the identity is given by the stabilizers of finite tuples. (The intuition is that two permutations are close together if they agree on a large finite subset of the domain.) This topology is derived from either of the following two metrics. (We identify the permutation domain with $\mathbb{N}$.)

(a) $d(g, g) = 0$ and $d(g, h) = 1/2^n$ where n is such that $ig = ih$ for $i < n$ but $ng \neq nh$, for $g \neq h$.

(b) $d'(g, g) = 0$ and $d'(g, h) = 1/2^n$ where n is such that $ig = ih$ and $ig^{-1} = ih^{-1}$ for $i < n$ but either $ng \neq nh$ or $ng^{-1} \neq nh^{-1}$, for $g \neq h$.

(Here $1/2^n$ could be replaced by any decreasing sequence tending to zero.) The advantage of the metric d' is that the symmetric group is a complete metric space for this metric. (The cycles $(1, 2, \ldots, n)$ form a Cauchy sequence for d which does not converge to a permutation.)

The closed and open subgroups of the symmetric group can be characterised as follows. A *first-order structure* consists of a set carrying a collection of relations and functions (of various positive arities) and constants; it is *relational* if there are no

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978-0-521-69469-8 - Groups St Andrews 2005, Volume 1
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functions or constants. Thus, a graph is an example of a relational structure (with a single binary relation). Just as for graphs, a relational structure M is *homogeneous* if any isomorphism between finite induced substructures can be extended to an automorphism of M.

Theorem 2.8 (a) *A subgroup H of the symmetric group is open if and only if it contains the stabiliser of a finite tuple.*

(b) *A subgroup H of the symmetric group is closed if and only if it is the full automorphism group of some first-order structure M; this structure can be taken to be a homogeneous relational structure.*

The first part of this theorem gives an interpretation of the small index property:

Proposition 2.9 *A permutation group G of countable degree has the small index property if and only if any subgroup of G with index less than $2^{\aleph_0}$ is open in G.*

Thus, if a closed group has the small index property, then its topology can be recovered from its group structure: the subgroups of index less than $2^{\aleph_0}$ form a basis of open neighbourhoods of the identity.

For the second part of the theorem, we define the *canonical relational structure* associated with a permutation group G as follows: For each orbit O of G on n-tuples of points of the domain Ω, we take an n-ary relation R_O, and specify that R_O holds precisely for those n-tuples which belong to the orbit O. The resulting structure is easily seen to be homogeneous. Now the closure of G is the automorphism group of its canonical relational structure.

A subgroup H of G is dense in G if and only if G and H have the same orbits on Ω^n for all $n \in \mathbb{N}$. In particular, a subgroup H of $\mathrm{Aut}(R)$ is dense in $\mathrm{Aut}(R)$ if and only if it acts homogeneously on R (as described earlier). Thus, for example, the second theorem of Bhattacharjee and Macpherson asserts that $\mathrm{Aut}(R)$ has a locally finite dense subgroup.

A remarkable classification of closed subgroups of the symmetric group of countable degree has been given by Bergman and Shelah [5]. Call two subgroups G_1 and G_2 of $\mathrm{Sym}(\Omega)$ *equivalent* if there exists a finite subset U of $\mathrm{Sym}(\Omega)$ such that $\langle G_1 \cup U\rangle = \langle G_2 \cup U\rangle$.

Theorem 2.10 *If Ω is countable, there are just four equivalence classes of closed subgroups of* $\mathrm{Sym}(\Omega)$. *They are characterised by the following conditions, where $G_{(\Gamma)}$ denotes the pointwise stabiliser of the subset Γ:*

(a) *For every finite subset Γ of Ω, the subgroup $G_{(\Gamma)}$ has at least one infinite orbit on Ω.*

(b) *There exists a finite subset Γ such that all the orbits of $G_{(\Gamma)}$ are finite, but none such that the cardinalities of these orbits have a common upper bound.*

(c) *There exists a finite subset Γ such that the orbits of $G_{(\Gamma)}$ have a common upper bound, but none such that $G_{(\Gamma)} = 1$.*

(d) *There exists a finite subset Γ such that $G_{(\Gamma)} = 1$.*

Note that $\mathrm{Aut}(R)$ falls into class (a) of this theorem; in fact, for any finite set Γ, all the orbits of $G_{(\Gamma)}$ outside Γ are infinite, and have the property that the induced subgraph is isomorphic to R and $G_{(\Gamma)}$ acts homogeneously on it. (If $|W| = n$, then the pointwise stabiliser of W has 2^n orbits outside W: for each subset U of W, there is an orbit consisting of the points witnessing condition $(*)$ for the pair (U, V), where $V = W \setminus U$.) So there is a finite set B such that $\langle \mathrm{Aut}(R), B\rangle = \mathrm{Sym}(R)$. (This fact follows from Theorem 1.1 of [43]. Galvin [30] showed that we can take B to consist of a single element. As we will see in the final section of the paper, these results are not specific to $\mathrm{Aut}(R)$.)

Recall that a subset of a metric space is *residual* if it contains a countable intersection of dense open sets. The *Baire category theorem* states that a residual set in a complete metric space is non-empty. Indeed, residual sets are 'large'; their properties are analogous to those of sets of full measure in a measure space. (For example, the intersection of countably many residual sets is dense.)

Now let G denote any closed subgroup of the symmetric group of countable degree. The element $g \in G$ is said to be *generic* if its conjugacy class is residual in G. A given group has at most one conjugacy class of generic elements. There may be no such class: for example, if G is discrete, then the only residual set is G itself.

As an example, we show that a permutation having infinitely many cycles of each finite length and no infinite cycles is generic in the symmetric group. Obviously the set P of such permutations is a conjugacy class. It suffices to show that, for each n, the set P_n of permutations of $\mathbb{N}$ having at least n cycles of length i for $i = 1, \ldots, n$ and in which the point n lies in a finite cycle is open and dense: for the intersection of the sets P_n is obviously P.

If $g \in P_n$, then there is a finite set X_g such that any permutation agreeing with g on X_g is in P_n. This set is an open ball; so P_n is open.

Any open ball is defined by a finite partial permutation h of $\mathbb{N}$, and consists of all permutations which extend h. Let h be any finite partial permutation. Then there is a finite extension of h which is a permutation of a finite set. By adjoining some more cycles, we may assume that this extension has at least n cycles of length i for $1 \leq i \leq n$ and that n lies in a finite cycle. Thus, the open ball defined by h meets P_n. So P_n is dense.

The first part of the following theorem is due to Truss [65]; the second is due to Hodges *et al.* [34], and is crucial for proving the small index property for $\mathrm{Aut}(R)$.

Theorem 2.11 (a) *The group* $\mathrm{Aut}(R)$ *contains generic elements. Such an element has infinitely many cycles of any given finite length and no infinite cycles.*

(b) *For any positive integer n, the group* $\mathrm{Aut}(R)^n$ *contains generic elements. Such an n-tuple generates a free subgroup of* $\mathrm{Aut}(R)$*, all of whose orbits are finite.*

2.6 Reducts of R

We conclude this section with Thomas' theorem [60] about the reducts of R. For my purposes here, a *reduct* of a structure A is a structure B on the same set with $\mathrm{Aut}(B) \geq \mathrm{Aut}(A)$. If our structures are first-order, then their automorphism groups are closed subgroups of the symmetric group on A; so we are looking for closed overgroups of $\mathrm{Aut}(A)$.

An *anti-automorphism* of a graph Γ is a permutation of the vertices mapping Γ to its complement. A *switching automorphism* is a permutation mapping Γ to a graph equivalent to it under switching. (Here *switching* with respect to a set X of vertices consists in replacing edges between X and its complement by non-edges, and non-edges by edges, leaving edges within or outside X unchanged.) A *switching anti-automorphism* is a permutation mapping Γ to a graph equivalent to its complement under switching. Now Thomas' Theorem asserts:

Theorem 2.12 *The closed subgroups of* $\mathrm{Sym}(R)$ *containing* $\mathrm{Aut}(R)$ *are:*

- $\mathrm{Aut}(R)$*;*
- *the group of automorphisms and anti-automorphisms of R;*
- *the group of switching automorphisms of R;*
- *the group of switching automorphisms and anti-automorphisms of R;*
- $\mathrm{Sym}(R)$.

3 Homogeneous relational structures

3.1 Fraïssé's Theorem

For finite permutation groups, as is well known, a consequence of the Classification of Finite Simple Groups is the fact that the only finite 6-transitive permutation groups are the symmetric and alternating groups. What about the infinite case? Analogues of the "geometric" multiply-transitive finite permutation groups such as projective general linear groups give groups which are at most 3-transitive. Can we achieve higher degrees of transitivity?

I do not know any trivial construction which produces an infinite permutation group with any prescribed degree of transitivity. If we take Ω to be countable, and let g_i be any permutation fixing i points and permuting the others in a single cycle, then it is clear that $\langle g_0, \ldots, g_{n-1} \rangle$ is n-transitive; but it may be $(n+1)$-transitive, or even highly transitive. A sufficiently clever choice of the permutations might give a group which is not $(n+1)$-transitive. However, it is more straightforward to ensure that the group is not $(n+1)$-transitive by letting it preserve a suitable $(n+1)$-ary relation, and to choose this relation to have an n-transitive automorphism group.

A theorem of Fraïssé guarantees the existence of suitable structures. I now describe this theorem.

We work in the context of relational structures over a fixed relational language (that is, the relations are named, and relations with the same name in different structures have the same arity; moreover, the induced substructure on a subset of