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Stark–Heegner points and special values of L-series

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Introduction

Let E be an elliptic curve over $\mathbb{Q}$ attached to a newform f of weight two on $\Gamma_0(N)$. Let K be a real quadratic field, and let $p\|N$ be a prime of multiplicative reduction for E which is inert in K, so that the p-adic completion K_p of K is the quadratic unramified extension of $\mathbb{Q}_p$.

Subject to the condition that all the primes dividing $M := N/p$ are split in K, the article [Dar] proposes an analytic construction of "Stark–Heegner points" in $E(K_p)$, and conjectures that these points are defined over specific class fields of K. More precisely, let

$$R := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}[1/p]) \text{ such that } M \text{ divides } c \right\}$$

be an Eichler $\mathbb{Z}[1/p]$-order of level M in $M_2(\mathbb{Q})$, and let $\Gamma := R_1^\times$ denote the group of elements in R of determinant 1. This group acts by Möbius transformations on the K_p-points of the p-adic upper half-plane

$$\mathcal{H}_p := \mathbb{P}^1(K_p) - \mathbb{P}^1(\mathbb{Q}_p),$$

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and preserves the non-empty subset $\mathcal{H}_p \cap K$. In [Dar], modular symbols attached to f are used to define a map

$$\Phi : \Gamma\backslash(\mathcal{H}_p \cap K) \longrightarrow E(K_p), \tag{0.1}$$

whose image is conjectured to consist of points defined over ring class fields of K. Underlying this conjecture is a more precise one, analogous to the classical Shimura reciprocity law, which we now recall.

Given $\tau \in \mathcal{H}_p \cap K$, the collection $\mathcal{O}_\tau$ of matrices $g \in R$ satisfying

$$g\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda_g \begin{pmatrix} \tau \\ 1 \end{pmatrix} \quad \text{for some } \lambda_g \in K, \tag{0.2}$$

is isomorphic to a $\mathbb{Z}[1/p]$-order in K, via the map $g \mapsto \lambda_g$. This order is also equipped with the attendant ring homomorphism $\eta : \mathcal{O}_\tau \longrightarrow \mathbb{Z}/M\mathbb{Z}$ sending g to its upper left-hand entry (taken modulo M). The map η is sometimes referred to as the *orientation* at M attached to τ. Conversely, given any $\mathbb{Z}[1/p]$-order $\mathcal{O}$ of discriminant prime to M equipped with an orientation η, the set $\mathcal{H}_p^{\mathcal{O}}$ of $\tau \in \mathcal{H}_p$ with associated oriented order equal to $\mathcal{O}$ is preserved under the action of Γ, and the set of orbits $\Gamma\backslash\mathcal{H}_p^{\mathcal{O}}$ is equipped with a natural simply transitive action of the group $G = \mathrm{Pic}^+(\mathcal{O})$, where $\mathrm{Pic}^+(\mathcal{O})$ denotes the narrow Picard group of oriented projective $\mathcal{O}$-modules of rank one. Denote this action by $(\sigma, \tau) \mapsto \tau^\sigma$, for $\sigma \in G$ and $\tau \in \Gamma\backslash\mathcal{H}_p^{\mathcal{O}}$. Class field theory identifies G with the Galois group of the *narrow ring class field* of K attached to $\mathcal{O}$, denoted H_K. It is conjectured in [Dar] that the points $\Phi(\tau)$ belong to $E(H_K)$ for all $\tau \in \mathcal{H}_p^{\mathcal{O}}$, and that

$$\Phi(\tau)^\sigma = \Phi(\tau^\sigma), \qquad \text{for all } \sigma \in \mathrm{Gal}(H_K/K) = \mathrm{Pic}^+(\mathcal{O}). \tag{0.3}$$

In particular it is expected that the point

$$P_K := \Phi(\tau_1) + \cdots + \Phi(\tau_h)$$

should belong to $E(K)$, where $\tau_1, \ldots, \tau_h$ denote representatives for the distinct orbits in $\Gamma\backslash\mathcal{H}_p^{\mathcal{O}}$. The article [BD3] shows that the image of P_K in $E(K_p) \otimes \mathbb{Q}$ is of the form $t \cdot \mathbf{P}_K$, where

(i) t belongs to $\mathbb{Q}^\times$;
(ii) $\mathbf{P}_K \in E(K)$ is of infinite order precisely when $L'(E/K, 1) \neq 0$;

provided the following ostensibly extraneous assumptions are satisfied

(i) $\bar{P}_K = a_p P_K$, where $\bar{P}_K$ is the Galois conjugate of P_K over K_p, and a_p is the pth Fourier coefficient of f.
(ii) The elliptic curve E has at least two primes of multiplicative reduction.

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The main result of [BD3] falls short of being definitive because of these two assumptions, and also because it only treats the image of P_K modulo the torsion subgroup of $E(K_p)$.

The main goal of this article is to examine certain "finer" invariants associated to P_K and to relate these to special values of L-series, guided by the analogy between the point P_K and classical Heegner points attached to imaginary quadratic fields.

In setting the stage for the main formula, let $E/\mathbb{Q}$ be an elliptic curve of conductor M; it is essential to assume that all the primes dividing M are *split* in K. This hypothesis is very similar to the one imposed in [GZ] when K is imaginary quadratic, where it implies that $L(E/K, 1)$ vanishes systematically because the sign in its functional equation is -1. In the case where K is real quadratic the "Gross-Zagier hypothesis" implies that the sign in the functional equation for $L(E/K, s)$ is 1 so that $L(E/K, s)$ vanishes to even order and is expected to be frequently non-zero at $s = 1$. Consistent with this expectation is the fact that the Stark–Heegner construction is now unavailable, in the absence of a prime $p \| M$ which is inert in K.

The main idea is to bring such a prime into the picture by "raising the level at p" to produce a newform g of level $N = Mp$ which is *congruent* to f. The congruence is modulo an appropriate ideal λ of the ring $\mathcal{O}_g$ generated by the Fourier coefficients of g. Let A_g denote the abelian variety quotient of $J_0(N)$ attached to g by the Eichler-Shimura construction. The main objective, which can now be stated more precisely, is to relate the *local behaviour at p* of the Stark–Heegner points in $A_g(K_p)$ to the algebraic part of the special value of $L(E/K, 1)$, taken modulo λ.

The first key ingredient in establishing such a relationship is an extension of the map Φ of (0.1) to arbitrary eigenforms of weight 2 on $\Gamma_0(Mp)$ such as g, and not just eigenforms with rational Fourier coefficients attached to elliptic curves, in a precise enough form so that phenomena related to congruences between modular forms can be analyzed. Let $\mathbb{T}$ be the full algebra of Hecke operators acting on the space of forms of weight two on $\Gamma_0(Mp)$. The theory presented in Section 1, based on the work of the third author [Das], produces a torus T over K_p equipped with a natural $\mathbb{T}$-action, whose character group (tensored with $\mathbb{C}$) is isomorphic as a $\mathbb{T} \otimes \mathbb{C}$-module to the space of weight 2 modular forms on $\Gamma_0(Mp)$ which are new at p. It also builds a Hecke-stable lattice $L \subset T(K_p)$, and a map Φ generalising (0.1)

$$\Phi : \Gamma \backslash (\mathcal{H}_p \cap K) \longrightarrow T(K_p)/L. \qquad (0.4)$$

It is conjectured in Section 1 that the quotient T/L is isomorphic to the rigid analytic space associated to an abelian variety J defined over $\mathbb{Q}$. A strong

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partial result in this direction is proven in [Das], where it is shown that T/L is isogenous over K_p to the rigid analytic space associated to the p-new quotient $J_0(N)^{p\text{-new}}$ of the jacobian $J_0(N)$. In Section 1, it is further conjectured that the points $\Phi(\tau) \in J(K_p)$ satisfy the same algebraicity properties as were stated for the map Φ of (0.1).

Letting Φ_p denote the group of connected components in the Néron model of J over the maximal unramified extension of $\mathbb{Q}_p$, one has a natural Hecke-equivariant projection

$$\partial_p : J(\mathbb{C}_p) \longrightarrow \Phi_p. \tag{0.5}$$

The group Φ_p is described explicitly in Section 1, yielding a concrete description of the Hecke action on Φ_p and a description of the primes dividing the cardinality of Φ_p in terms of "primes of fusion" between forms on $\Gamma_0(M)$ and forms on $\Gamma_0(Mp)$ which are new at p.

This description also makes it possible to attach to E and K an explicit element

$$\mathcal{L}(E/K,1)_{(p)} \in \bar{\Phi}_p,$$

where $\bar{\Phi}_p$ is a suitable f-isotypic quotient of Φ_p. Thanks to a theorem of Popa [Po], this element is closely related to the special value $L(E/K,1)$, and, in particular, one has the equivalence

$$L(E/K,1) = 0 \quad \Longleftrightarrow \quad \mathcal{L}(E/K,1)_{(p)} = 0 \text{ for all } p.$$

Section 2 contains an exposition of Popa's formula.

Section 3 is devoted to a discussion of $\mathcal{L}(E/K,1)_{(p)}$; furthermore, by combining the results of Sections 1 and 2, it proves the main theorem of this article, an avatar of the Gross-Zagier formula which relates Stark–Heegner points to special values of L-series.

Main Theorem. *For all primes p which are inert in K,*

$$\partial_p(P_K) = \mathcal{L}(E/K,1)_{(p)}.$$

Potential arithmetic applications of this theorem (conditional on the validity of the deep conjectures of Section 1) are briefly discussed in Section 4.

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1 Stark–Heegner points on $J_0(Mp)^{p\text{-new}}$

Heegner points on an elliptic curve E defined over $\mathbb{Q}$ can be defined analytically by certain complex line integrals involving the modular form

$$f := \sum_{n=1}^{\infty} a_n(E) e^{2\pi i n z}$$

corresponding to E, and the Weierstrass parametrization of E. To be precise, let τ be any point of the complex upper half plane $\mathcal{H} := \{z \in \mathbb{C} | \Im z > 0\}$. The complex number

$$J_\tau := \int_\infty^\tau 2\pi i f(z) dz \in \mathbb{C}$$

gives rise to an element of $\mathbb{C}/\Lambda_E \cong E(\mathbb{C})$, where Λ_E is the Néron lattice of E, and hence to a complex point $P_\tau \in E(\mathbb{C})$. If τ also lies in an imaginary quadratic subfield K of $\mathbb{C}$, then P_τ is a *Heegner point* on E. The theory of complex multiplication shows that this analytically defined point is actually defined over an abelian extension of K, and it furthermore prescribes the action of the Galois group of K on this point.

The Stark–Heegner points of [Dar], defined on elliptic curves over $\mathbb{Q}$ with multiplicative reduction at p, are obtained by replacing complex integration on $\mathcal{H}$ with a double integral on the product of a p-adic and a complex upper half plane $\mathcal{H}_p \times \mathcal{H}$.

We now very briefly describe this construction. Let E be an elliptic curve over $\mathbb{Q}$ of conductor $N = Mp$, with $p \nmid M$. The differential $\omega := 2\pi i f(z) dz$ and its anti-holomorphic counterpart $\bar{\omega} = -2\pi i f(\bar{z}) d\bar{z}$ give rise to two elements in the DeRham cohomology of $X_0(N)(\mathbb{C})$:

$$\omega^\pm := \omega \pm \bar{\omega}.$$

To each of these differential forms is attached a *modular symbol*

$$m_E^\pm\{x \to y\} := (\Omega_E^\pm)^{-1} \int_x^y \omega^\pm, \qquad \text{for } x, y \in \mathbb{P}^1(\mathbb{Q}).$$

Here $\Omega_E^\pm$ is an appropriate complex period chosen so that $m_E^\pm$ takes values in $\mathbb{Z}$ and in no proper subgroup of $\mathbb{Z}$.

The group Γ defined in the Introduction acts on $\mathbb{P}^1(\mathbb{Q}_p)$ by Möbius transformations. For each pair of cusps $x, y \in \mathbb{P}^1(\mathbb{Q})$ and choice of sign $\pm$, a $\mathbb{Z}$-valued additive measure $\mu^\pm\{x \to y\}$ on $\mathbb{P}^1(\mathbb{Q}_p)$ can be defined by

$$\mu^\pm\{x \to y\}(\gamma \mathbb{Z}_p) = m_E^\pm\{\gamma^{-1}x \to \gamma^{-1}y\}, \tag{1.1}$$

where γ is an element of Γ. Since the stabilizer of $\mathbb{Z}_p$ in Γ is $\Gamma_0(N)$, equation (1.1) is independent of the choice of γ by the $\Gamma_0(N)$-invariance of $m_E^\pm$. The

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motivation for this definition, and a proof that it extends to an additive measure on $\mathbb{P}^1(\mathbb{Q}_p)$, comes from "spreading out" the modular symbol $m_E^{\pm}$ along the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$ (see [Dar], [Das], and Section 1.2 below). For any $\tau_1, \tau_2 \in \mathcal{H}_p$ and $x, y \in \mathbb{P}^1(\mathbb{Q}_p)$, a multiplicative double integral on $\mathcal{H}_p \times \mathcal{H}$ is then defined by (multiplicatively) integrating the function $(t-\tau_1)/(t-\tau_2)$ over $\mathbb{P}^1(\mathbb{Q}_p)$ with respect to the measure $\mu^{\pm}\{x \to y\}$:

$$\begin{aligned} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_x^y \omega_{\pm} &:= \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu^{\pm}\{x \to y\}(t) \\ &= \lim_{||\mathcal{U}|| \to 0} \prod_{U \in \mathcal{U}} \left(\frac{t_U - \tau_2}{t_U - \tau_1}\right)^{\mu^{\pm}\{x \to y\}(U)} \in \mathbb{C}_p^{\times}. \end{aligned} \tag{1.2}$$

Here the limit is taken over uniformly finer disjoint covers $\mathcal{U}$ of $\mathbb{P}^1(\mathbb{Q}_p)$ by open compact subsets U, and t_U is an arbitrarily chosen point of U. Choosing special values for the limits of integration, in a manner motivated by the classical Heegner construction described above, one produces special elements in $\mathbb{C}_p^{\times}$. These elements are transferred to E using Tate's p-adic uniformization $\mathbb{C}_p^{\times}/q_E \cong E(\mathbb{C}_p)$ to define Stark–Heegner points.

In order to lift the Stark–Heegner points on E to the Jacobian $J_0(N)^{p\text{-new}}$, one can replace the modular symbols attached to E with the universal modular symbol for $\Gamma_0(N)$. In this section, we review this construction of Stark–Heegner points on $J_0(N)^{p\text{-new}}$, as described in fuller detail in [Das].

1.1 The universal modular symbol for $\Gamma_0(N)$

The first step is to generalize the measures $\mu^{\pm}\{x \to y\}$ on $\mathbb{P}^1(\mathbb{Q}_p)$. As we will see, the new measure naturally takes values in the p-new quotient of the homology group $H_1(X_0(N), \mathbb{Z})$. Once this measure is defined, the construction of Stark–Heegner points on $J_0(N)^{p\text{-new}}$ can proceed as the construction of Stark–Heegner points on E given in [Dar]. The Stark–Heegner points on $J_0(N)^{p\text{-new}}$ will map to those on E under the modular parametrization $J_0(N)^{p\text{-new}} \to E$.

We begin by recalling the universal modular symbol for $\Gamma_0(N)$. Let $\mathcal{M} := \mathrm{Div}_0\, \mathbb{P}^1(\mathbb{Q})$ be the group of degree zero divisors on the set of cusps of the complex upper half plane, defined by the exact sequence

$$0 \to \mathcal{M} \to \mathrm{Div}\, \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Z} \to 0. \tag{1.3}$$

The group Γ acts on $\mathcal{M}$ via its action on $\mathbb{P}^1(\mathbb{Q})$ by Möbius transformations.

For any abelian group G, a *G-valued modular symbol* is a homomorphism $m : \mathcal{M} \longrightarrow G$; we write $m\{x \to y\}$ for $m([x]-[y])$. Let $\mathcal{M}(G)$ denote the

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left Γ-module of G-valued modular symbols, where the action of Γ is defined by the rule

$$(\gamma m)\{x \to y\} = m\{\gamma^{-1}x \to \gamma^{-1}y\}.$$

Note that the natural projection onto the group of coinvariants

$$\mathcal{M} \longrightarrow \mathcal{M}_{\Gamma_0(N)} = H_0(\Gamma_0(N), \mathcal{M})$$

is a $\Gamma_0(N)$-invariant modular symbol. Furthermore, this modular symbol is universal, in the sense that any other $\Gamma_0(N)$-invariant modular symbol factors through this one.

One can interpret $H_0(\Gamma_0(N), \mathcal{M})$ geometrically as follows. Given a divisor $[x]-[y] \in \mathcal{M}$, consider any path from x to y in the completed upper half plane $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. Identifying the quotient $\Gamma_0(N)\backslash(\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))$ with $X_0(N)(\mathbb{C})$, this path gives a well-defined element of $H_1(X_0(N), \text{cusps}, \mathbb{Z})$, the singular homology of the Riemann surface $X_0(N)(\mathbb{C})$ relative to the cusps. Manin [Man] proves that this map induces an isomorphism between the maximal torsion-free quotient $H_0(\Gamma_0(N), \mathcal{M})_T$ and $H_1(X_0(N), \text{cusps}, \mathbb{Z})$. Furthermore, the torsion of $H_0(\Gamma_0(N), \mathcal{M})$ is finite and supported at 2 and 3. The projection

$$\mathcal{M} \to \mathcal{M}_{\Gamma_0(N)} \to H_1(X_0(N), \text{cusps}, \mathbb{Z})$$

is called the *universal modular symbol for* $\Gamma_0(N)$.

The points of $X_0(N)$ over $\mathbb{C}$ correspond to isomorphism classes of pairs (E, C_N) of (generalized) elliptic curves $E/\mathbb{C}$ equipped with a cyclic subgroup $C_N \subset E$ of order N. To such a pair we can associate two points of $X_0(M)$, namely the points corresponding to the pairs (E, C_M) and $(E/C_p, C_N/C_p)$, where C_p and C_M are the subgroups of C_N of size p and M, respectively. This defines two morphisms of curves

$$f_1 : X_0(N) \to X_0(M) \text{ and } f_2 : X_0(N) \to X_0(M), \tag{1.4}$$

each of which is defined over $\mathbb{Q}$. The map f_2 is the composition of f_1 with the Atkin-Lehner involution W_p on $X_0(N)$. Write $f_* = f_{1*} \oplus f_{2*}$ and $f^* = f_1^* \oplus f_2^*$ (resp. $\overline{f_*}$ and $\overline{f^*}$) for the induced maps on singular homology (resp. relative singular homology):

$$\begin{aligned}
f_* :&\quad H_1(X_0(N), \mathbb{Z}) \to H_1(X_0(M), \mathbb{Z})^2\\
\overline{f_*} :&\quad H_1(X_0(N), \text{cusps}, \mathbb{Z}) \to H_1(X_0(M), \text{cusps}, \mathbb{Z})^2\\
f^* :&\quad H_1(X_0(M), \mathbb{Z})^2 \to H_1(X_0(N), \mathbb{Z})\\
\overline{f^*} :&\quad H_1(X_0(M), \text{cusps}, \mathbb{Z})^2 \to H_1(X_0(N), \text{cusps}, \mathbb{Z}).
\end{aligned}$$

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The abelian variety $J_0(N)^{p\text{-new}}$ is defined to be the quotient of $J_0(N)$ by the images of the Picard maps on Jacobians associated to f_1 and f_2. Define $\overline{H}$ and H to be the maximal torsion-free quotients of the cokernels of $\overline{f^*}$ and f^*, respectively:

$$\overline{H} := (\mathrm{Coker}\,\overline{f^*})_T \text{ and } H := (\mathrm{Coker}\, f^*)_T.$$

If we write g for the dimension of $J_0(N)^{p\text{-new}}$, the free abelian groups $\overline{H}$ and H have ranks $2g+1$ and $2g$, respectively, and the natural map $H \to \overline{H}$ is an injection ([Das, Prop. 3.2]).

The groups H and $\overline{H}$ have Hecke actions generated by T_ℓ for $\ell \nmid N$, U_ℓ for $\ell | N$, and W_p. We omit the proof of the following proposition.

Proposition 1.1 *The group* $(\overline{H}/H)_T \cong \mathbb{Z}$ *is Eisenstein; that is,* T_ℓ *acts as* $\ell+1$ *for* $\ell \nmid N$, U_ℓ *acts as* ℓ *for* $\ell | M$, *and* W_p *acts as* -1.

Proposition 1.1 implies that it is possible to choose a Hecke equivariant map $\psi : \overline{H} \to H$ such that the composition

$$H \longrightarrow \overline{H} \xrightarrow{\psi} H \tag{1.5}$$

has finite cokernel. For example, we may take ψ to be the Hecke operator $(p^2-1)(T_r - (r+1))$ for any prime $r \nmid N$. We fix a choice of ψ for the remainder of the paper.

1.2 A p-adic uniformization of $J_0(N)^{p\text{-new}}$

For any free abelian group G, let $\mathrm{Meas}(\mathbb{P}^1(\mathbb{Q}_p), G)$ denote the Γ-module of G-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ with total measure zero, where Γ acts by $(\gamma\mu)(U) := \mu(\gamma^{-1}U)$.

In order to construct a Γ-invariant $\mathrm{Meas}(\mathbb{P}^1(\mathbb{Q}_p), H)$-valued modular symbol, we recall the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(\mathbb{Q}_p)$. The set of vertices $\mathcal{V}(\mathcal{T})$ of $\mathcal{T}$ is identified with the set of homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. Two vertices v and v' are said to be adjacent if they can be represented by lattices L and L' such that L contains L' with index p. Let $\mathcal{E}(\mathcal{T})$ denote the set of oriented edges of $\mathcal{T}$, that is, the set of ordered pairs of adjacent vertices of $\mathcal{T}$. Given $e = (v_1, v_2)$ in $\mathcal{E}(\mathcal{T})$, call $v_1 = s(e)$ the source of e, and $v_2 = t(e)$ the target of e. Define the standard vertex v^o to be the class of $\mathbb{Z}_p^2$, and the standard oriented edge $e^o = (v^o, v)$ to be the edge whose source is v^o and whose stabilizer in Γ is equal to $\Gamma_0(N)$. Note that $\mathcal{E}(\mathcal{T})$ is equal to the disjoint union of the Γ-orbits of e^o and $\bar{e}^o$, where $\bar{e}^o = (v, v^o)$ is the opposite edge of e^o. A *half line* of $\mathcal{T}$ is a sequence (e_n) of oriented edges such that $t(e_n) = s(e_{n+1})$. Two half lines are said to be equivalent if they have in common all but a finite

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number of edges. It is known that the boundary $\mathbb{P}^1(\mathbb{Q}_p)$ of the p-adic upper half plane bijects onto the set of equivalence classes of half lines. For an oriented edge e, write U_e for the subset of $\mathbb{P}^1(\mathbb{Q}_p)$ whose elements correspond to classes of half lines passing through e. The sets U_e are determined by the rules: (1) $U_{\bar{e}^o} = \mathbb{Z}_p$, (2) $U_{\bar{e}} = \mathbb{P}^1(\mathbb{Q}_p) - U_e$, and (3) $U_{\gamma e} = \gamma U_e$ for all $\gamma \in \Gamma$. The U_e give a covering of $\mathbb{P}^1(\mathbb{Q}_p)$ by compact open sets. Finally, recall the existence of a Γ-equivariant reduction map

$$r : (K_p - \mathbb{Q}_p) \longrightarrow \mathcal{V}(\mathcal{T}),$$

defined on the K_p-points of $\mathcal{H}_p$. (As before, K_p is an unramified extension of $\mathbb{Q}_p$.) See [GvdP] for more details.

Define a function

$$\kappa\{x \to y\} : \mathcal{E}(\mathcal{T}) \longrightarrow H$$

as follows. When e belongs to the Γ-orbit of e^o and $\gamma \in \Gamma$ is chosen so that $\gamma e = e^o$, let $\kappa\{x \to y\}(e)$ be ψ applied to the image of $\gamma^{-1}([x] - [y])$ in $\overline{H}$ under the universal modular symbol for $\Gamma_0(N)$. Let $\kappa\{x \to y\}(e)$ be the negative of this value when the relation $\gamma e = \bar{e}^o$ holds.

The function $\kappa\{x \to y\}$ is a *harmonic cocycle on* $\mathcal{T}$, that is, it obeys the rules

(i) $\kappa\{x \to y\}(\bar{e}) = -\kappa\{x \to y\}(e)$ for all $e \in \mathcal{E}(\mathcal{T})$, and
(ii) $\sum_{s(e)=v} \kappa\{x \to y\}(e) = 0$ for all $v \in \mathcal{V}(\mathcal{T})$, where the sum is taken over the $p+1$ oriented edges e whose source $s(e)$ is v.

Furthermore, we have the Γ-invariance property

$$\kappa\{\gamma x \to \gamma y\}(\gamma e) = \kappa\{x \to y\}(e)$$

for all $\gamma \in \Gamma$.

The natural bijection between $\mathrm{Meas}(\mathbb{P}^1(\mathbb{Q}_p), H)$ and the group of harmonic cocycles on $\mathcal{T}$ valued in H shows that the definition

$$\mu\{x \to y\}(U_e) := \kappa\{x \to y\}(e)$$

yields a Γ-invariant $\mathrm{Meas}(\mathbb{P}^1(\mathbb{Q}_p), H)$-valued modular symbol μ ([Das, Prop. 3.1]). When $m = [x] - [y] \in \mathcal{M}$, we write μ_m for $\mu\{x \to y\}$.

We can now define, for $\tau_1, \tau_2 \in \mathcal{H}_p$ and $m \in \mathcal{M}$, a multiplicative double integral attached to the universal modular symbol for $\Gamma_0(N)$:

$$\begin{aligned} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_m \omega \quad &:= \quad \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t - \tau_2}{t - \tau_1} \right) d\mu_m(t) \\ &= \lim_{||\mathcal{U}|| \to 0} \prod_{U \in \mathcal{U}} \left(\frac{t_U - \tau_2}{t_U - \tau_1} \right) \otimes \mu_m(U) \in \mathbb{C}_p^\times \otimes_{\mathbb{Z}} H, \end{aligned}$$

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with notations as in (1.2). One shows that this integral is Γ-invariant:

$$\times\!\!\!\!\!\!\int_{\gamma\tau_1}^{\gamma\tau_2}\int_{\gamma m}\omega = \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2}\int_{m}\omega \quad \text{for } \gamma \in \Gamma.$$

Letting T denote the torus $T = \mathbb{G}_m \otimes_{\mathbb{Z}} H$, we thus obtain a homomorphism

$$\begin{aligned} ((\mathrm{Div}_0\,\mathcal{H}_p)\otimes\mathcal{M})_\Gamma &\to T \\ ([\tau_1]-[\tau_2])\otimes m &\mapsto \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2}\int_{m}\omega. \end{aligned} \tag{1.6}$$

Consider the short exact sequence of Γ-modules defining $\mathrm{Div}_0\,\mathcal{H}_p$:

$$0 \to \mathrm{Div}_0\,\mathcal{H}_p \to \mathrm{Div}\,\mathcal{H}_p \to \mathbb{Z} \to 0.$$

After tensoring with $\mathcal{M}$, the long exact sequence in homology gives a boundary map

$$\delta_1 : H_1(\Gamma, \mathcal{M}) \to ((\mathrm{Div}_0\,\mathcal{H}_p)\otimes\mathcal{M})_\Gamma. \tag{1.7}$$

The long exact sequence in homology associated to the sequence (1.3) defining $\mathcal{M}$ gives a boundary map

$$\delta_2 : H_2(\Gamma, \mathbb{Z}) \to H_1(\Gamma, \mathcal{M}). \tag{1.8}$$

Define L to be the image of $H_2(\Gamma, \mathbb{Z})$ under the composed homomorphisms in (1.6), (1.7), and (1.8): $H_2(\Gamma, \mathbb{Z}) \to T(\mathbb{Q}_p)$. Note that the Hecke algebra $\mathbb{T}$ of H acts on T.

Theorem 1.2 ([Das], Thm. 3.3) *Let K_p denote the quadratic unramified extension of $\mathbb{Q}_p$. The group L is a discrete, Hecke stable subgroup of $T(\mathbb{Q}_p)$ of rank $2g$. The quotient T/L admits a Hecke-equivariant isogeny over K_p to the rigid analytic space associated to the product of two copies of $J_0(N)^{p\text{-new}}$.*

Remark 1.3 If one lets the nontrivial element of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$ act on T/L by the Hecke operator U_p, the isogeny of Theorem 1.2 is defined over $\mathbb{Q}_p$.

Remark 1.4 As described in [Das, §5.1], Theorem 1.2 is a generalization of a conjecture of Mazur, Tate, and Teitelbaum [MTT, Conjecture II.13.1] which was proven by Greenberg and Stevens [GS].

Theorem 1.2 implies that T/L is isomorphic to the rigid analytic space associated to an abelian variety J defined over a number field (which can be embedded in $\mathbb{Q}_p$). We now state a conjectural refinement of Theorem 1.2.

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