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The Problems

1. The Lion and the Christian. A lion and a Christian in a closed circular Roman arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal? In other words, can the lion catch the Christian in finite time?

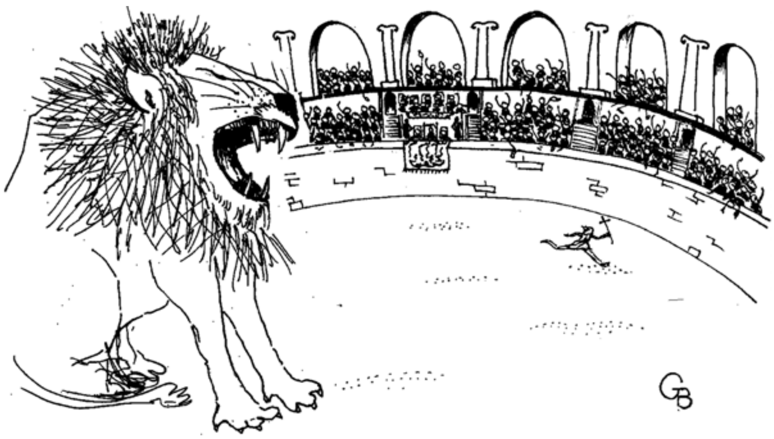

Fig. 1. A Roman lion.

2. Integer Sequences

(i) Show that among $n+1$ positive integers none of which is greater than $2n$ there are two such that one divides the other.

(ii) Show that among $n+1$ positive integers none of which is greater than $2n$ there are two that are relatively prime.

(iii) Suppose that we have n natural numbers none of which is greater than $2n$ such that the least common multiple of any two is greater than $2n$. Show that all n numbers are greater than $2n/3$.

(iv) Show that every sequence of $n = rs + 1$ distinct integers with $r, s \geq 1$ has an increasing subsequence of length $r + 1$ or a decreasing subsequence of length $s + 1$.

3. Points on a Circle

(i) Let X and Y be subsets of the vertex set of a regular n-gon. Show that there is a rotation ϱ of this polygon such that $|X \cap \varrho(Y)| \geq |X||Y|/n$, where, as usual, $|Z|$ denotes the number of elements in a finite set Z.

(ii) Let $S = F_1 \cup F_2$, where F_1 and F_2 are closed subsets of S, the unit circle in $\mathbb{R}^2$. Show that either F_1 or F_2 is such that, for every angle $0 < \alpha \leq \pi$, it contains two points that form angle α with the centre.

(iii) A set S of integers is *sum-free* if $x+y = z$ has no solution in S, i.e. $x+y \notin S$ whenever $x, y \in S$. Show that every set of $n \geq 1$ non-zero integers contains a sum-free subset of size greater than $n/3$.

4. Partitions into Closed Sets. Can the plane be partitioned into countably many non-empty closed sets? And what about $\mathbb{R}^n$?

5. Triangles and Squares. What is the maximal area of a triangle contained in a unit square? And the minimal area of a triangle containing a unit square?

6. Polygons and Rectangles. Show that every convex polygon of area 1 is contained in a rectangle of area 2.

7. African Rally. A car circuit goes through the towns $T_1, T_2, \ldots, T_n$ in this cyclic order (so from T_i the route leads to T_{i+1} and from T_n back to T_1), and a car is to travel around this circuit, starting from one of the towns. At the start of the journey, the tank of the car is empty, but in each town T_i it can pick up p_i amount of fuel.

(i) Show that if $\sum_{i=1}^n p_i$ is precisely sufficient to drive round the entire circuit then there is a town such that if the car starts from there then it can complete the entire circuit without running out of fuel.

(ii) Show that if each section of the circuit (from town T_i to T_{i+1}) needs an integer amount of fuel, each p_i is an integer *and* $\sum_{i=1}^n p_i$ is precisely 1 more than the amount of fuel needed to drive round the entire course, then there is precisely one town such that, starting from there, there is at least 1 unit of fuel in the tank not only at the end but throughout the circuit.

8. Fixing Convex Domains. A convex board is surrounded by some nails hammered into a table: the nails make impossible to slide the board in any direction, but if any of them is missing then this is no longer true. What is the maximal number of nails?

9. Nested Subsets. Is an infinite family of *nested* subsets of a countable set necessarily countable?

10. Almost Disjoint Subsets. Call two sets *almost disjoint* if their intersection is finite. Is there an uncountable family of almost disjoint subsets of a countable set?

11. Loaded Dice. We have two loaded dice, with $1, 2, \ldots, 6$ coming up with various (possibly different) probabilities on each. Is it possible that when we roll them both, each of the sums $2, 3, \ldots, 12$ comes up with the same probability?

12. An Unexpected Inequality. Let $a_1, a_2, \ldots$ be positive reals. Show that

$$\limsup_{n\to\infty} \left(\frac{1+a_{n+1}}{a_n}\right)^n \geq e.$$

Show also that the inequality need not hold with e replaced by a larger number.

13. Colouring Lines. Let $\mathcal{L}$ be a collection of k-element subsets of a set X. We call the elements of X *points*, and the k-subsets belonging to $\mathcal{L}$ *lines*. 'Maker' and 'Breaker' play a game by alternately colouring the points of X *red* and *blue*, with Breaker making the first move. Once a point is coloured, it is never recoloured. Maker aims to 'make' a *red line*, a line all whose points are coloured red, and Breaker wants to prevent this. Show that if there are $2^k - 1$ lines then Breaker has a winning strategy, but for some arrangement of 2^k lines Maker has a winning strategy. What are the corresponding numbers if Maker makes the first move?

14. Independent Sets. Let G be a graph with vertex set V, and write $d(v)$ for the degree of a vertex v. Show that G contains at least

$$\sum_{v\in V} \frac{1}{d(v)+1}$$

independent vertices.

15. Expansion into Sums $2^i 3^j$. Every natural number may be expressed in binary form, i.e., is the sum of some numbers $1, 2, 2^2, 2^3, \ldots$. In every such expansion, of any two of the summands one divides the other. Is it possible to write every natural number as a sum of numbers of the form $2^i 3^j$ such that no summand divides the other? It is easily checked that the first few numbers have such representations; for example, $19 = 4 + 6 + 9$, $23 = 6 + 8 + 9$, $115 = 16 + 27 + 72$.

16. A Tennis Match. Alice and Bob are about to play a tennis match consisting of a single set. They decide that Alice will serve in the first game, and the first to reach twelve games wins the match (whether two games ahead or not). However, they are considering two serving schemes: the *alternating serves* scheme, in which the servers alternate game by game (how surprising!), and the *winner serves* scheme, in which the winner of a game serves in the next.

Fig. 2. The big question.

Alice estimates that she has 0.71 chance of winning her serve, while Bob has only 0.67 chance to hold serve. Which scheme should Alice choose to maximize her chances of winning?

17. A Triangle Inequality. Let R be the circumradius of an acute-angled triangle, r its inradius and h the length of the longest height. Show that $r + R \leq h$. When does equality hold?

18. Planar Domains of Diameter 1. Show that every planar domain D of diameter 1 is contained in a regular hexagon of width 1, i.e., side-length $\sqrt{3}/3$.

Deduce that every planar domain of diameter 1 can be partitioned into three sets of diameter at most $\sqrt{3}/2$.

19. Orienting Graphs. Show that for every graph there is an orientation of the edges such that for every vertex the out-degree and in-degree differ by at most 1.

20. A Simple Clock. How many times a day is it impossible to tell the time by a clock with identical hour and minute hands, provided we can always tell whether it is a.m. or p.m.?

21. Neighbours in a Matrix. Show that every $n \times n$ matrix whose entries are $1, 2, \ldots, n^2$ in some order has two neighbouring entries (in a row or in a column) that differ by at least n.

22. Separately Continuous Functions. Let $f : S = [0, 1]^2 \to \mathbb{R}$ be separately continuous in its variables, i.e., continuous in x for every fixed y, and continuous in y for every fixed x. Show that if $f^{-1}(0)$ is dense in the square S then it is identically 0.

23. Boundary Cubes. A down-set in the solid d-dimensional cube $[0, n]^d \subset \mathbb{R}^d$ is a set $D \subset [0, n]^d$ such that if $0 \le x_i \le y_i \le n$ for $i = 1, \ldots, n$, and $\mathbf{y} = (y_i)_1^d \in D$ then $\mathbf{x} = (x_i)_1^d \in D$ as well. What is the maximal number of boundary integral unit cubes, i.e., solid unit cubes

$$[c_1 - 1, c_1] \times [c_2 - 1, c_2] \times \cdots \times [c_d - 1, c_d],$$

with each c_i an integer, $1 \le c_i \le n$, that meets both a down-set D and its complement $\overline{D}$?

24. Lozenge Tilings. A *lozenge* or *calisson* is a rhombus of side-length 1, with angles $\pi/3$ and $2\pi/3$, i.e., the union of two equilateral triangles of side-length 1 that share a side. Figure 3 shows a tiling of a regular hexagon with lozenges.

Show that no matter how we tile a regular hexagon with lozenges, we must use the same number of tiles of each orientation.

25. A Continuum Independent Set. To every real number x, assign a finite set $\Phi(x) \subset \mathbb{R} \setminus \{x\}$. Call a set $S \subset \mathbb{R}$ *independent* if $x \in \Phi(y)$ does not hold for $x, y \in S$, that is if $S \cap \Phi(S) = \emptyset$. Show that there is an independent set $S \subset \mathbb{R}$ whose cardinality is that of the continuum.

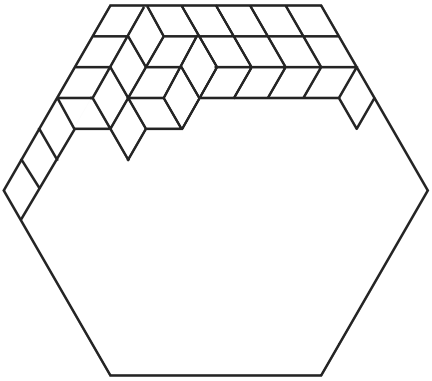

Fig. 3. The beginning of a tiling of the regular hexagon with lozenges.

26. Separating Families of Sets. Given an n-element set S, what is the minimal number of subsets $S_1, \ldots, S_k$ of S needed to separate all pairs of elements of S, i.e., so that for all $x, y \in S$ there is a set S_i containing one of x and y, but not the other?

Equivalently, at least how many bipartite graphs are needed to cover (all the edges of) a complete graph on n vertices?

27. Bipartite Covers of Complete Graphs. Suppose that a family of bipartite graphs covers all the edges of a complete graph K_n. Show that altogether there are at least $n \log_2 n$ vertices in these bipartite graphs.

More precisely, show that if $n = 2^k + \ell < 2^{k+1}$, $\ell \geq 0$, then the minimum is precisely $nk + 2\ell$.

28. Convexity and Intersecting Simplices

(i) Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n+2}\}$ be a set of $n + 2$ points in $\mathbb{R}^n$ and, for a non-empty subset I of $\{1, \ldots, n+2\}$, let $X(I)$ be the convex hull of the points $\mathbf{x}_i$, $i \in I$. Show that there are disjoint sets I, J such that $X(I) \cap X(J) \neq \emptyset$.

(ii) The convex hull conv$X$ of a set $X \subset \mathbb{R}^n$ is the smallest convex set containing X: the intersection of all convex sets containing it. Equivalently,

$$\operatorname{conv}X = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \,:\, \mathbf{x}_i \in X,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1,\ k = 1, 2, \ldots \right\}.$$

Show that in the sums above we need not take more than $n+1$ terms, i.e., the convex hull of a set $X \subset \mathbb{R}^n$ is

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{n+1} \lambda_i \mathbf{x}_i : \mathbf{x}_i \in X, \ \lambda_i \geq 0, \ \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

29. Intersecting Convex Sets. Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^n$ such that for $k \leq n+1$ any k members of $\mathcal{C}$ have a non-empty intersection. Show that the intersection of *all* members of $\mathcal{C}$ is non-empty.

30. Judicious Partitions of Points. Let $P_1, \ldots, P_n$ be n points in the plane. Show that there is a point P such that every line through P has at least $n/3$ points P_i in each of the two closed half-planes it determines.

31. Further Lozenge Tilings. Consider a hexagon in which every angle is $2\pi/3$ and the side-lengths are $a_1, \ldots, a_6$. For what values of $a_1, \ldots, a_6$ is there a tiling of this hexagon with *'lozenges'* of the three possible orientations, as in Problem 24; see Figure 4. How many lozenges are there of each orientation?

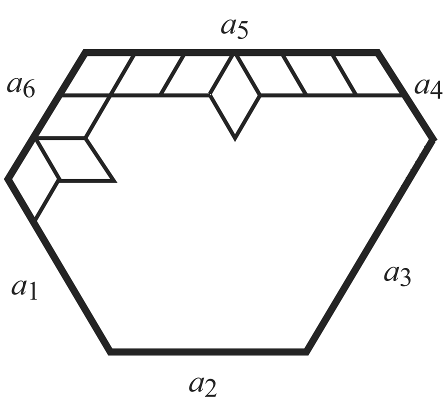

Fig. 4. A hexagon $H(a)$, with the beginning of a lozenge tiling.

32. Two Squares in a Square. Place two squares (with disjoint interiors) into a unit square. Show that the sum of the side-lengths is at most 1.

33. Lines Through Points. Let n points (of a Euclidean space) have the property that the line joining any two of them passes through a third point of the set. Must the n points all lie on one line?

34. The Spread of Infection on a Square Grid. A disease spreads on an n by n grid G_n as follows. At time 0 some of the *sites* (vertices, lattice points, grid points) are *infected*, the others are *healthy* but *susceptible*. Infected sites remain infected for ever, and at every time t, $t = 1, 2, \ldots$, every healthy site with at least two infected neighbours becomes itself infected. More formally, at time $t = 0$ we have a set S_0 of infected sites, and at each time step t, $t = 1, 2, \ldots$, we let S_t be the set of sites with at least two neighbours in S_{t-1}, together with the sites in S_{t-1}. S_t is the set of sites infected at time t. A set S_0 of initially infected sites is said to *percolate* on the *board* G_n, with 2-neighbour infection or parameter 2, if, for some t, S_t is the entire set of sites, i.e., if, eventually, every site becomes infected, as in Figure 5. What is the minimal number of initially infected sites that percolate?

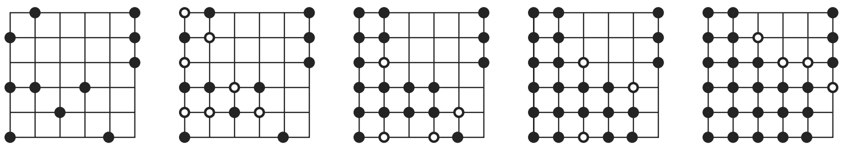

Fig. 5. The spread of a disease on a 6 by 6 grid, with the newly infected sites denoted by open circles ∘. At time $t = 0$ we have 11 infected sites, and at time 4 there are 29. In another two steps all sites become infected.

35. The Spread of Infection in a d-dimensional Box. Let us consider the following extension of Problem 34. This time the disease spreads in a d-dimensional $n \times n \times \cdots \times n$ box

$$B = [n]^d = \{(x_i)_1^d \in \mathbb{Z}^d : 1 \le x_i \le n, i = 1, \ldots, d\}$$

with n^d sites. As in Problem 34, the disease starts with a set S of infected sites in B. Every site with at least two infected neighbours becomes infected, and so is ready to infect other (neighbouring) sites. What is the minimal number of sites in S if eventually every site in B becomes infected?

36. Sums of Integers. Let $1 \le a_1 < a_2 < \cdots < a_\ell \le n$ be integers with $\ell > (n+1)/2$. Show that $a_i + a_j = a_k$ for some $1 \le i < j < k \le \ell$.

37. Normal Numbers. Given a real number

$$x = 0.x_1 x_2 \cdots = \sum_{i=1}^{\infty} x_i 10^{-i}$$

written in base 10, and a sequence (word)

$$w = w_1 w_2 \ldots w_k = \sum_{i=1}^{k} w_i 10^{k-i},$$

with terms (letters) 0, 1, ..., 9, write $f_n(w; x)$ for the number of times $x_1x_2\dots x_n$ *contains* w, i.e., the number of suffices i, $0 \le i \le n-k$, such that $k_j = w_{i+j}$ for $j = 1, \dots, k$. Note that if the letters of w are chosen at random (with every number 0, 1, ..., 9 having probability 1/10) then the expectation of $f_n(w; x)$ is $(n-k+1)10^{-k} \sim n10^{-k}$.

A real number is *normal in base* 10 if for every k-letter word $w = w_1w_2\dots w_k$ with letters 0, 1, ..., 9, we have $\lim_{n\to\infty} f_n(w; x)/n = 10^{-k}$.

Let $\gamma = 0.123\ \dots\ 9\,10\,11\ \dots\ 99\,100\,101\ \dots$ be the real number whose significant digits are formed by the concatenation of all the natural numbers. Show that γ is normal in base 10.

38. Random Walks on Graphs. Let s and t be vertices of a graph G. A random walk on G starts at s and stops at t. Show that the expected number of times this walk traverses the edges of a cycle in one direction is equal to the expected number of traversals in the other direction.

39. Simple Tilings of Rectangles. A tiling of a rectangle R with $n \ge 2$ rectangles is called *simple* if no rectangle strictly inside R is a union of at least two rectangles of the tiling. For what values of n is there a simple tiling of a rectangle? Also, for large values of n, at least how many essentially different rectangles are there?

40. *L*-tilings. Cut out a square of a 2^n by 2^n chess board. Show that the remaining $2^{2n} - 1$ squares can be tiled with L-tiles, where an *L-tile* is a union of three squares sharing a vertex, as the tiles in Figure 6.

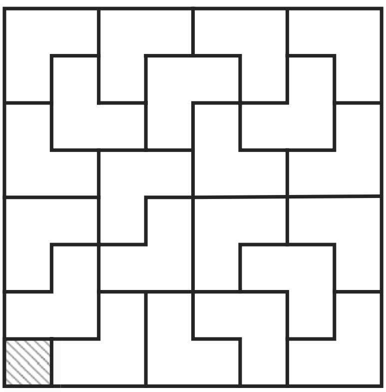

Fig. 6. An L-tiling of the 8×8 board with the bottom left square cut out.

41. Antipodal Points and Maps. For $D \subset \mathbb{R}^n$ and $R \subset \mathbb{R}^m$, a map $f : D \to R$ is said to be *odd* if $f(-x) = -f(x)$ for every $x \in D$. Two points x and y of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ are *antipodal* if $y = -x$. In view of this, an odd map $S^n \to S^m$ is also said to be *antipodal*, since it maps antipodal points into antipodal points. Borsuk proved that continuous antipodal maps $S^n \to S^n$ have odd degrees: in particular, they are not null-homotopic. This implies that

(1) there is no continuous antipodal map $S^n \to S^{n-1}$.

Show that (1) is equivalent to each of the following three statements.

(2) Every continuous map $S^n \to \mathbb{R}^n$ sends at least one pair of antipodal points into the same point.
(3) Every continuous odd map $S^n \to \mathbb{R}^n$ sends at least one point (and so at least one pair of antipodal points) into the origin of $\mathbb{R}^n$.
(4) For every family of $n + 1$ closed sets covering S^n, one of the sets contains a pair of antipodal points.

42. Bodies of Diameter 1. Let $K \subset \mathbb{R}^d$ be a body of diameter 1. Show that K is contained in a (closed) ball of radius $r = \sqrt{d/(2d+2)}$, and deduce that it can be partitioned into $2^{d-1} + 1$ sets, each of diameter strictly less than 1.

43. Equilateral Triangles. Show that if equilateral triangles are erected externally (or internally) on the sides of a triangle as in Figure 7 then the centres of the new triangles form an equilateral triangle whose centroid is the centroid of the original triangle.

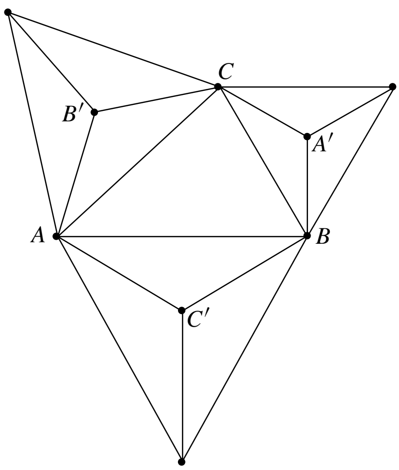

Fig. 7. Three equilateral triangles erected on a triangle ABC, with centroids A', B' and C'.