Affine embeddings of homogeneous spaces

Ivan V. Arzhantsev

Introduction

Throughout the paper G denotes a connected reductive algebraic group, unless otherwise specified, and H an algebraic subgroup of G. All groups and algebraic varieties considered are over an algebraically closed field $\mathbb{K}$ of characteristic zero, unless otherwise specified. Let $\mathbb{K}[X]$ be the algebra of regular functions on an algebraic variety X and $\mathbb{K}(X)$ the field of rational functions on X provided X is irreducible. Our general references are [30] for algebraic groups and [56, 37, 29] for algebraic transformation groups and invariant theory.

Affine embeddings: definitions. Let us recall that an irreducible algebraic G-variety X is said to be an *embedding* of the homogeneous space G/H if X contains an open G-orbit isomorphic to G/H. We shall denote this relationship by $G/H \hookrightarrow X$. Let us say that an embedding $G/H \hookrightarrow X$ is *affine* if the variety X is affine. In many problems of invariant theory, representation theory and other branches of mathematics, only affine embeddings of homogeneous spaces arise. This is why it is reasonable to study specific properties of affine embeddings in the framework of a well-developed general embedding theory.

Which homogeneous spaces admit an affine embedding? It is easy to show that a homogeneous space G/H admits an affine embedding if and only if G/H is quasi-affine (as an algebraic variety). In this situation, the subgroup H is said to be *observable* in G. A closed subgroup H of G is observable if and only if there exist a rational finite-dimensional G-module V and a vector $v \in V$ such that the stabilizer G_v coincides with H. (This follows from the fact that any affine G-variety may be realized as a closed invariant subvariety in a finite-dimensional G-module [56, Th.1.5].) There is a nice group-theoretic description of

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observable subgroups due to A. Sukhanov: a subgroup H is observable in G if and only if there exists a quasi-parabolic subgroup $Q \subset G$ such that $H \subset Q$ and the unipotent radical H^u is contained in the unipotent radical Q^u, see [63], [29, Th.7.3]. (Let us recall that a subgroup Q is said to be *quasi-parabolic* if Q is the stabilizer of a highest weight vector in some G-module V.)

It follows from Chevalley's theorem that any subgroup H without non-trivial characters (in particular, any unipotent subgroup) is observable. By Matsushima's criterion, a homogeneous space G/H is affine if and only if H is reductive. (For a simple proof, see [42] or [4]; a characteristic-free proof can be found in [57].) In particular, any reductive subgroup is observable. A description of affine homogeneous spaces G/H for non-reductive G is still an open problem.

Complexity of reductive group actions. Now we define the notion of complexity, which we shall encounter many times in the text. Let us fix the notation. By $B = TU$ denote a Borel subgroup of G with a maximal torus T and the unipotent radical U. By definition, the *complexity* $c(X)$ of a G-variety X is the codimension of a B-orbit of general position in X for the restricted action $B : X$. This notion firstly appeared in [45] and [70]. Now it plays a central role in embedding theory. By Rosenlicht's theorem, $c(X)$ is equal to the transcendence degree of the field $\mathbb{K}(X)^B$ of rational B-invariant functions on X. A normal G-variety X is called *spherical* if $c(X) = 0$ or, equivalently, $\mathbb{K}(X)^B = \mathbb{K}$. A homogeneous space G/H and a subgroup $H \subseteq G$ are said to be *spherical* if G/H is a spherical G-variety.

Rational representations, the isotypic decomposition and G-algebras. A linear action of G in vector space W is said to be *rational* if for any vector $w \in W$ the linear span $\langle Gw \rangle$ is finite-dimensional and the action $G : \langle Gw \rangle$ defines a representation of an algebraic group. Since any finite-dimensional representation of G is completely reducible, it is easy to prove that W is a direct sum of finite-dimensional simple G-modules.

Let $\Xi_+(G)$ be the semigroup of dominant weights of G. For any $\lambda \in \Xi_+(G)$, denote by W_λ the sum of all simple submodules in W of highest weight λ. The subspace W_λ is called an *isotypic component* of W of weight λ, and the decomposition

$$W = \oplus_{\lambda \in \Xi_+(G)} W_\lambda$$

is called *the isotypic decomposition* of W.

If G acts on an affine variety X, the linear action $G : \mathbb{K}[X]$, $(gf)(x) := f(g^{-1}x)$, is rational [56, Lemma 1.4]. (Note that for irreducible X the action on rational functions $G : \mathbb{K}(X)$ defined by the same formula is not rational.) The isotypic decomposition

$$\mathbb{K}[X] = \oplus_{\lambda \in \Xi_+(G)} \mathbb{K}[X]_\lambda$$

and its interaction with the multiplicative structure on $\mathbb{K}[X]$ give important technical tools for the study of affine embeddings.

An affine G-variety X is spherical if and only if $\mathbb{K}[X]_\lambda$ is either zero or a simple G-module for any $\lambda \in \Xi_+(G)$ [32].

Suppose that $\mathfrak{A}$ is a commutative associative algebra with unit over $\mathbb{K}$. If G acts on $\mathfrak{A}$ by automorphisms and the action $G : \mathfrak{A}$ is rational, we say that $\mathfrak{A}$ is a *G-algebra*. The algebra $\mathbb{K}[X]$ is a G-algebra for any affine G-variety X. Moreover, any finitely generated G-algebra without nilpotents arises in this way.

We conclude the introduction with a review of the contents of this survey.

One of the pioneering works in embedding theory was a classification of normal affine $SL(2)$-embeddings due to V. L. Popov, see [52, 37]. In the same period (early seventies) the theory of toric varieties was developed. A toric variety may be considered as an equivariant embedding of an algebraic torus T. Such embeddings are described in terms of convex fans. Any cone in the fan of a toric variety X represents an affine toric variety. This reflects the fact that X has a covering by T-invariant affine charts. In 1972, V. L. Popov and E. B. Vinberg [55] described affine embeddings of quasi-affine homogeneous spaces G/H, where H contains a maximal unipotent subgroup of G. In Section 1 we discuss briefly these results together with a more recent one: a remarkable classification of algebraic monoids with a reductive group G as the group of invertible elements (E. B. Vinberg [71]). This is precisely the classification of affine embeddings of the space $(G \times G)/\Delta(G)$, where $\Delta(G)$ is the diagonal subgroup.

In Section 2 we consider connections of the theory of affine embeddings with Hilbert's 14th problem. Let H be an observable subgroup of G. By the Grosshans theorem, the following conditions are equivalent:

1) the algebra of invariants $\mathbb{K}[V]^H$ is finitely generated for any G-module V;
2) the algebra of regular functions $\mathbb{K}[G/H]$ is finitely generated;

3) there exists a (normal) affine embedding $G/H \hookrightarrow X$ such that

$$\operatorname{codim}_X(X \setminus (G/H)) \geq 2$$

(such an embedding is called *the canonical embedding* of G/H).

It was proved by F. Knop that if $c(G/H) \leq 1$ then the algebra $\mathbb{K}[G/H]$ is finitely generated. This result provides a large class of subgroups with a positive solution of Hilbert's 14th problem. In particular, Knop's theorem together with Grosshans' theorem on the unipotent radical P^u of a parabolic subgroup $P \subset G$ includes almost all known results on Popov-Pommerening's conjecture (see 2.2). We study the canonical embedding of G/P^u from a geometric view-point. Finally, we mention counterexamples to Hilbert's 14th problem due to M. Nagata, P. Roberts, and R. Steinberg.

In Section 3 we introduce the notion of *an affinely closed space*, i.e. an affine homogeneous space admitting no non-trivial affine embeddings, and discuss the result of D. Luna related to this notion. (We say that an affine embedding $G/H \hookrightarrow X$ is *trivial* if $X = G/H$.) Affinely closed spaces of an arbitrary affine algebraic group are characterized and some elementary properties of affine embeddings are formulated.

Section 4 is devoted to affine embeddings with a finite number of orbits. We give a characterization of affine homogeneous spaces G/H such that any affine embedding of G/H contains a finite number of orbits. More generally, we compute the maximal number of parameters in a continuous family of G-orbits over all affine embeddings of a given affine homogeneous space G/H. The group of equivariant automorphisms of an affine embedding is also studied here.

Some applications of the theory of affine embeddings to functional analysis are given in Section 5. Let $M = K/L$ be a homogeneous space of a connected compact Lie group K, and $C(M)$ the commutative Banach algebra of all complex-valued continuous functions on M. The K-action on $C(M)$ is defined by the formula $(kf)(x) = f(k^{-1}x)$, $k \in K$, $x \in M$. We shall say that A is an *invariant algebra* on M if A is a K-invariant uniformly closed subalgebra with unit in $C(M)$. Denote by G and H the complexifications of K and L respectively. Then G is a reductive algebraic group with reductive subgroup H. There exists a correspondence between finitely generated invariant algebras on M and affine embeddings of G/F with some additional data, where F is an observable subgroup of G containing H. This correspondence was introduced by V. M. Gichev [25], I. A. Latypov [38, 39] and, in a more algebraic way, by E. B.

Vinberg. We give a precise formulation of this correspondence and reformulate some facts on affine embeddings in terms of invariant algebras. Some results of this section are new and not published elsewhere.

The last section is devoted to G-algebras. It is easy to prove that any subalgebra in the polynomial algebra $\mathbb{K}[x]$ is finitely generated. On the other hand, one can construct many non-finitely generated subalgebras in $\mathbb{K}[x_1, \ldots, x_n]$ for $n \geq 2$. More generally, every subalgebra in an associative commutative finitely generated integral domain $\mathfrak{A}$ with unit is finitely generated if and only if $\operatorname{Kdim} \mathfrak{A} \leq 1$, where $\operatorname{Kdim} \mathfrak{A}$ is the Krull dimension of $\mathfrak{A}$ (Proposition 6.5). In Section 6 we obtain an equivariant version of this result. The problem was motivated by the study of invariant algebras in the previous section. The proof of the main result (Theorem 6.3) is based on a geometric method for constructing a non-finitely generated subalgebra in a finitely generated G-algebra and on properties of affine embeddings obtained above. In particular, the notion of an affinely closed space is crucial for the classification of G-algebras with finitely generated invariant subalgebras. The arguments used in this text are slightly different from the original ones [9]. A characterization of G-algebras with finitely generated invariant subalgebras for non-reductive G is also given in this section.

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1 Remarkable classes of affine embeddings

1.1 Affine toric varieties

We begin with some notation. Let T be an algebraic torus and $\Xi(T)$ the lattice of its characters. A T-action on an affine variety X defines a $\Xi(T)$-grading on the algebra $\mathbb{K}[X] = \oplus_{\chi \in \Xi(T)} \mathbb{K}[X]_\chi$, where $\mathbb{K}[X]_\chi = \{f \mid tf = \chi(t)f \text{ for any } t \in T\}$. (This grading is just the isotypic decomposition, see the introduction.) If X is irreducible, then the set $L(X) = \{\chi \mid \mathbb{K}[X]_\chi \neq 0\}$ is a submonoid in $\Xi(T)$.

Definition 1.1. An *affine toric variety* X is a normal affine T-variety with an open T-orbit isomorphic to T.

Below we list some basic properties of T-actions:

- An action $T : X$ has an open orbit if and only if $\dim \mathbb{K}[X]_\chi = 1$ for any $\chi \in L(X)$. In this situation $\mathbb{K}[X]$ is T-isomorphic to the semigroup algebra $\mathbb{K}L(X)$.
- An action $T : X$ is effective if and only if the subgroup in $\Xi(T)$ generated by $L(X)$ coincides with $\Xi(T)$.
- Suppose that $T : X$ is an effective action with an open orbit. Then the following conditions are equivalent:

1) X is normal;
2) the semigroup algebra $\mathbb{K}L(X)$ is integrally closed;
3) if $\chi \in \Xi(T)$ and there exists $n \in \mathbb{N}$, $n > 0$, such that $n\chi \in L(X)$, then $\chi \in L(X)$ (the saturation condition);
4) there exists a solid convex polyhedral cone K in $\Xi(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $L(X) = K \cap \Xi(T)$.

In this situation, any T-invariant radical ideal of $\mathbb{K}[X]$ corresponds to the subsemigroup $L(X) \setminus M$ for a fixed face M of the cone K. This correspondence defines a bijection between T-invariant radical ideals of $\mathbb{K}[X]$ and faces of K.

The proof of these properties can be found, for example, in [23]. Summarizing all the results, we obtain

Theorem 1.2. *Affine toric varieties are in one-to-one correspondence with solid convex polyhedral cones in the space $\Xi(T) \otimes_{\mathbb{Z}} \mathbb{Q}$; and T-orbits on a toric variety are in one-to-one correspondence with faces of the cone.*

The classification of affine toric varieties will serve us as a sample for studying more complicated classes of affine embeddings. Generalizations of a combinatorial description of toric varieties were obtained for spherical varieties [45, 33, 18], and for embeddings of complexity one [68]. In this more general context, the idea that normal G-varieties may be described by some convex cones becomes rigorous through the method of U-invariants developed by D. Luna and Th. Vust. The essence of this method is contained in the following theorem (see [72, 37, 54, 29]).

Theorem 1.3. *Let $\mathfrak{A}$ be a G-algebra and U a maximal unipotent subgroup of G. Consider the following properties of an algebra:*

- *it is finitely generated;*
- *it has no nilpotent elements;*
- *it has no zero divisors;*
- *it is integrally closed.*

If (P) is any of these properties, then the algebra $\mathfrak{A}$ *has property (P) if and only if the algebra* $\mathfrak{A}^U$ *has property (P).*

We try to demonstrate briefly some applications of the method of U-invariants in the following subsections.

1.2 Normal affine $SL(2)$-embeddings

Suppose that the group $SL(2)$ acts on a normal affine variety X and there is a point $x \in X$ such that the stabilizer of x is trivial and the orbit $SL(2)x$ is open in X. We say in this case that X is a *normal* $SL(2)$*-embedding*.

Let U_m be a finite extension of the standard maximal unipotent subgroup in $SL(2)$:

$$U_m = \left\{ \begin{pmatrix} \epsilon & a \\ 0 & \epsilon^{-1} \end{pmatrix} \mid \ \epsilon^m = 1, \ a \in \mathbb{K} \right\}.$$

Theorem 1.4 ([52]). *Normal non-trivial* $SL(2)$*-embeddings are in one-to-one correspondence with rational numbers* $h \in (0, 1]$*. Furthermore,*

- $h = 1$ *corresponds to a (unique) smooth* $SL(2)$*-embedding with two orbits:* $X = SL(2) \cup SL(2)/T$*;*
- *if* $h = \frac{p}{q} < 1$ *and* $(p, q) = 1$*, then* $X = SL(2) \cup SL(2)/U_{p+q} \cup \{pt\}$*, and* $\{pt\}$ *is an isolated singular point in* X*.*

The proof of Theorem 1.4 can be found in [52], [37, Ch. 3]. Here we give only some examples and explain what the number h (which is called the *height* of X) means in terms of the algebra $\mathbb{K}[X]$.

Example 1.5. 1) The group $SL(2)$ acts tautologically on the space $\mathbb{K}^2$ and by conjugation on the space $\mathrm{Mat}(2 \times 2)$. Consider the point

$$x = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \in \mathrm{Mat}(2 \times 2) \times \mathbb{K}^2$$

and its orbit

$$SL(2)x = \{(A, v) \mid \det A = -1, \mathrm{tr}\ A = 0, Av = v, v \neq 0\}.$$

It is easy to see that the closure

$$X = \overline{SL(2)x} = \{(A, v) \mid \det A = -1, \operatorname{tr} A = 0, Av = v\}$$

is a smooth $SL(2)$-embedding with two orbits, hence X corresponds to $h = 1$.

2) Let $V_d = \langle x^d, x^{d-1}y, \ldots, y^d\rangle$ be the $SL(2)$-module of binary forms of degree d. It is possible to check that

$$X = \overline{SL(2)(x, x^2y)} \subset V_1 \oplus V_3$$

is a normal $SL(2)$-embedding with the orbit decomposition $X = SL(2) \cup SL(2)/U_3 \cup \{pt\}$, hence X corresponds to $h = \frac{1}{2}$.

An embedding $SL(2) \hookrightarrow X$, $g \to gx$ determines the injective homomorphism $\mathfrak{A} = \mathbb{K}[X] \to \mathbb{K}[SL(2)]$ with $Q\mathfrak{A} = Q\mathbb{K}[SL(2)]$, where $Q\mathfrak{A}$ is the quotient field of $\mathfrak{A}$. Let U^- be the unipotent subgroup of $SL(2)$ opposite to U. Then

$$\begin{aligned}\mathbb{K}[SL(2)]^{U^-} &= \{f \in \mathbb{K}[SL(2)] \mid f(ug) = f(g),\ g \in SL(2), u \in U^-\}\\ &= \mathbb{K}[A, B],\end{aligned}$$

where $A\begin{pmatrix} a & b \\ c & d\end{pmatrix} = a$ and $B\begin{pmatrix} a & b \\ c & d\end{pmatrix} = b$.

Below we list some facts ([37, Ch. 3]) that allow us to introduce the height of an $SL(2)$-embedding X.

- If $\mathfrak{C}$ is an integral F-domain, where F is a unipotent group, then $Q(\mathfrak{C}^F) = (Q\mathfrak{C})^F$. In particular, if $\mathfrak{C} \subseteq \mathfrak{A}$ and $Q\mathfrak{A} = Q\mathfrak{C}$, then $Q(\mathfrak{A}^{U^-}) = Q(\mathfrak{C}^{U^-})$.
- Suppose that $\lim_{t\to 0}\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} x$ exists. Then $A \in \mathbb{K}[SL(2)] \subset \mathbb{K}(X)$ is regular on X.
- Let $\mathfrak{D} \subset \mathbb{K}[x, y]$ be a homogeneous integrally closed subalgebra in the polynomial algebra such that $Q\mathfrak{D} = \mathbb{K}(x, y)$ and $x \in \mathfrak{D}$. Then $\mathfrak{D}$ is generated by monomials.

  In our situation, the algebra $\mathfrak{D} = \mathfrak{A}^{U^-} \subset \mathbb{K}[A, B]$ is homogeneous because it is T-stable (since T normalizes U^-).
- There exists rational $h \in (0, 1]$ such that

$$\mathfrak{A}^{U^-} = \mathfrak{A}(h) = \langle A^iB^j \mid \frac{j}{i} \le h\rangle.$$

  Moreover, for any rational $h \in (0, 1]$ the subspace $\langle SL(2)\mathfrak{A}(h)\rangle \subset \mathbb{K}[SL(2)]$ is a subalgebra.

Remark . While normal $SL(2)$-embeddings are parametrized by a discrete parameter h, there are families of non-isomorphic non-normal $SL(2)$-embeddings over a base of arbitrary dimension [13].

Remark . A classification of $SL(2)$-actions on normal three-dimensional affine varieties without open orbit can be found in [6, 5].

1.3 HV-varieties and S-varieties

In this subsection we discuss the results of V. L. Popov and E. B. Vinberg [55]. Throughout G denotes a connected and simply connected semisimple group.

Definition 1.6. An *HV-variety* X is the closure of the orbit of a highest weight vector in a simple G-module.

Let $V(\lambda)$ be the simple G-module with highest weight λ and v_λ a highest weight vector in $V(\lambda)$. Denote by λ^* the highest weight of the dual G-module $V(\lambda)^*$.

- $X(\lambda) = \overline{Gv_{\lambda^*}}$ is a normal affine variety consisting of two orbits: $X(\lambda) = Gv_{\lambda^*} \cup \{0\}$.
- $\mathbb{K}[X(\lambda)] = \mathbb{K}[Gv_{\lambda^*}] = \oplus_{m\geq 0}\mathbb{K}[X(\lambda)]_{m\lambda}$, any isotypic component $\mathbb{K}[X(\lambda)]_{m\lambda}$ is a simple G-module, and

$$\mathbb{K}[X(\lambda)]_{m_1\lambda}\mathbb{K}[X(\lambda)]_{m_2\lambda} = \mathbb{K}[X(\lambda)]_{(m_1+m_2)\lambda}.$$

- The algebra $\mathbb{K}[X(\lambda)]$ is a unique factorization domain if and only if λ is a fundamental weight of G.

Example 1.7. 1) The quadratic cone $KQ_n = \{x \in \mathbb{K}^n \mid x_1^2 + \cdots + x_n^2 = 0\}$ $(n \geq 3)$ is an HV-variety for the tautological representation $SO(n) : \mathbb{K}^n$. (In fact, the group $SO(n)$ is not simply connected and we consider the corresponding module as a Spin(n)-module.) It follows that KQ_n is normal and it is factorial if and only if $n \geq 5$.

2) The Grassmannian cone $KG_{n,m}$ $(n \geq 2,\ 1 \leq m \leq n-1)$ (i.e. the cone over the projective variety of m-subspaces in $\mathbb{K}^n$) is an HV-variety associated with the fundamental $SL(n)$-representation in the space $\bigwedge^m \mathbb{K}^n$, hence it is factorial.

Definition 1.8. An irreducible affine variety X with an action of a connected reductive group G is said to be an *S-variety* if X has an open G-orbit and the stabilizer of a point in this orbit contains a maximal unipotent subgroup of G.

Any S-variety may be realized as $X = \overline{Gv}$, where $v = v_{\lambda_1^*} + \cdots + v_{\lambda_k^*}$ is a sum of highest weight vectors $v_{\lambda_i^*}$ in some G-module V. We have the isotypic decomposition

$$\mathbb{K}[X] = \oplus_{\lambda \in L(X)} \mathbb{K}[X]_\lambda,$$

where $L(X)$ is the semigroup generated by $\lambda_1, \ldots, \lambda_k$, any $\mathbb{K}[X]_\lambda$ is a simple G-module, and $\mathbb{K}[X]_\lambda \mathbb{K}[X]_\mu = \mathbb{K}[X]_{\lambda+\mu}$. The last condition determines uniquely (up to G-isomorphism) the multiplicative structure on the G-module $\mathbb{K}[X]$. This shows that there is a bijection between S-varieties and finitely generated submonoids in $\Xi_+(G)$.

Consider the cone $K = \mathbb{Q}_+ L(X)$. As in the toric case, normality of X is equivalent to the saturation condition for the semigroup $L(X)$, and G-orbits on X are in one-to-one correspondence with faces of K. On the other hand, there are phenomena which are specific for S-varieties. For example, the complement to the open orbit in X has codimension ≥ 2 if and only if $\mathbb{Z}L(X) \cap \Xi_+(G) \subseteq \mathbb{Q}_+ L(X)$ (this is never the case for non-trivial toric varieties). Also, an S-variety X is factorial if and only if $L(X)$ is generated by fundamental weights.

Finally, we mention one more result on this subject. Say that an action $G : X$ on an affine variety X is *special* (or *horospherical*) if there is an open dense subset $W \subset X$ such that the stabilizer of any point of W contains a maximal unipotent subgroup of G.

Theorem 1.9 ([54]). *The following conditions are equivalent:*

- *the action $G : X$ is special;*
- *the stabilizer of any point on X contains a maximal unipotent subgroup;*
- $\mathbb{K}[X]_\lambda \mathbb{K}[X]_\mu \subseteq \mathbb{K}[X]_{\lambda+\mu}$ *for any* $\lambda, \mu \in \Xi_+(G)$.

1.4 Algebraic monoids

The general theory of algebraic semigroups was developed by M. S. Putcha, L. Renner and E. B. Vinberg. In this subsection we recall briefly the classification results following [71].

Definition 1.10. An *(affine) algebraic semigroup* is an (affine) algebraic variety S with an associative multiplication

$$\mu : \ S \times S \to S,$$

which is a morphism of algebraic varieties. An algebraic semigroup S is *normal* if S is a normal algebraic variety.