

Contents

<i>Preface</i>	<i>page ix</i>
1 Introduction	1
1.1 Ricci flow: what is it, and from where did it come?	1
1.2 Examples and special solutions	2
1.2.1 Einstein manifolds	2
1.2.2 Ricci solitons	3
1.2.3 Parabolic rescaling of Ricci flows	6
1.3 Getting a feel for Ricci flow	6
1.3.1 Two dimensions	6
1.3.2 Three dimensions	7
1.4 The topology and geometry of manifolds in low dimensions	11
1.5 Using Ricci flow to prove topological and geometric results	14
2 Riemannian geometry background	16
2.1 Notation and conventions	16
2.2 Einstein metrics	19
2.3 Deformation of geometric quantities as the Riemannian metric is deformed	20
2.3.1 The formulae	20
2.3.2 The calculations	23
2.4 Laplacian of the curvature tensor	31
2.5 Evolution of curvature and geometric quantities under Ricci flow	32

vi	<i>Contents</i>	
3	The maximum principle	35
3.1	Statement of the maximum principle	35
3.2	Basic control on the evolution of curvature	36
3.3	Global curvature derivative estimates	39
4	Comments on existence theory for parabolic PDE	43
4.1	Linear scalar PDE	43
4.2	The principal symbol	44
4.3	Generalisation to Vector Bundles	45
4.4	Properties of parabolic equations	47
5	Existence theory for the Ricci flow	48
5.1	Ricci flow is not parabolic	48
5.2	Short-time existence and uniqueness: The DeTurck trick	49
5.3	Curvature blow-up at finite-time singularities	52
6	Ricci flow as a gradient flow	55
6.1	Gradient of total scalar curvature and related functionals	55
6.2	The \mathcal{F} -functional	56
6.3	The heat operator and its conjugate	57
6.4	A gradient flow formulation	58
6.5	The classical entropy	61
6.6	The zeroth eigenvalue of $-4\Delta + R$	63
7	Compactness of Riemannian manifolds and flows	65
7.1	Convergence and compactness of manifolds	65
7.2	Convergence and compactness of flows	68
7.3	Blowing up at singularities I	69
8	Perelman's \mathcal{W} entropy functional	71
8.1	Definition, motivation and basic properties	71
8.2	Monotonicity of \mathcal{W}	76
8.3	No local volume collapse where curvature is controlled	79
8.4	Volume ratio bounds imply injectivity radius bounds	84
8.5	Blowing up at singularities II	86
9	Curvature pinching and preserved curvature properties under Ricci flow	88
9.1	Overview	88
9.2	The Einstein Tensor, E	88

<i>Contents</i>		vii
9.3	Evolution of E under the Ricci flow	89
9.4	The Uhlenbeck Trick	90
9.5	Formulae for parallel functions on vector bundles	93
9.6	An ODE-PDE theorem	95
9.7	Applications of the ODE-PDE theorem	98
10	Three-manifolds with positive Ricci curvature, and beyond	105
10.1	Hamilton's theorem	105
10.2	Beyond the case of positive Ricci curvature	107
A	Connected sum	108
	<i>References</i>	109
	<i>Index</i>	112